Towards a CAS promoting links between procedural and conceptual mathematical knowledge

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Promoting links between procedural and conceptual mathematical knowledge is an important goal of mathematics education that is by no means easy to achieve. As regards CAS-based mathematics education, such a state may partly be caused by some limitations of the available CAS environments. By examining a sample of expressions, functions, equations and inequalities, possible requirements for a CAS promoting links between the two knowledge types are identified thus suggesting how these environments may be refined in years to come.

1 INTRODUCTION

Computers can be used to change mathematics teaching by decreasing the time needed for procedural skills and increasing the time for conceptual understanding, the importance of which has been realised by many researchers. In many cases, a concept relies on a procedure, or vice versa. For example, the fact that a system of linear equations has an infinite number of solutions relies on a procedure confirming that one of its equations is equal to a linear combination of the others. Or, the fact that a correct (equivalent) transformation is applied to the given algebraic entity relies on some conceptual knowledge regarding the domains of the underlying functions. Although mathematics education should develop both procedural and conceptual knowledge and make links between the two, only a few CAL studies have examined the effects of their treatments regarding the co-ordination of procedural and conceptual mathematical knowledge. An in-depth study of this very important topic is therefore much needed, and it may take into account two recent papers of Haapasalo and Kadijevich. While Haapasalo & Kadijevich (2000) define relevant notions regarding the two knowledge types and examine the question of linking them at the theoretical and the instructional levels, Kadijevich & Haapasalo (2001) show that links between the two knowledge types

(Procedural-Conceptual links) can be established through learning activities requiring the production and use of rules as well as multiple representations, which have been implemented in two constructivist CAL environments developed by the authors themselves.

It is true that several CAS studies consider both procedural and conceptual knowledge (see, for example, Nurit, 1997), but, to our knowledge, only Hochfelsner & Kligner (1998) examined P-C links. They found that such links were not developed by the applied treatment. Bearing in mind Artigue's (2001) proposal that computational tools should be pedagogical instruments for promoting CAL learning, we find that, because of various procedural and conceptual limitations, the available CAS environments are not fully pedagogical instruments and thus some of the desired P-C links may not be established.

Having examined the symbolic mode of several CAS environments, we first list some observed limitations and then underline possible requirements for a CAS promoting P-C links, which suggest how these environments may be refined in years to come. Specifically, we will use a sample of tasks involving expressions, functions, equations and inequalities that are mostly relevant to tenth grade. Note that the reported limitations may not fully apply to a particular CAS environment, and while "CAS" denotes an existing CAS environment, "CAS+" denotes a desired elaboration of this environment.

2 LIMITATIONS

CAS may "know" that $\sqrt{xy}-\sqrt{x}\sqrt{y}$ is not equal to zero for all real values of x and y. It also may "know" that $\ln xy-(\ln x+\ln y)$ is not always equal to zero. However, a class of problems suggested by these examples - "Under what conditions is the given expression equal to zero?" - may not always be reliably solved with CAS. For example, CAS usually reinforces a common misconception that functions $f(x)=\frac{x^2}{x}, g(x)=(\sqrt{x})^2$ are equivalent on the whole real domain since it simplifies both $\frac{x^2}{x}-x$ and $(\sqrt{x})^2-x$ to zero. (This may not be the case with $\sqrt{x^2}-(\sqrt{x})^2$, but is the given answer $|x|-x$ still a correct one from the point of view of a tenth-grade student?) The same may hold true for the expressions $e^{\ln x}-x$ and $10^{\log x}-x$.

Our experience with four CASs revealed several examples of wrong or inadequate answers, even when variable domains were changed to subsets of R such as $[0,+\infty)$. A short summary of the observed fallacies is given below.

- CAS finds that $\frac{0}{x}=0$. For $\frac{x}{0}$, it returns "cannot divide by 0" (or $\pm\infty$). What then about $\frac{0}{0}$?

- If the user wants to simplify the equation $2x = 5x$ by dividing it by x, CAS does not protest and returns an absurd fact "2 = 5". This means that, armed with CAS, a less-able student may verify a number of "proofs" that "1 = 2". (CAS allows the user to simplify the equation $x(x-1) = x$ to $x - 1 = 1$, which is not equivalent to the initial one since the solution $x = 0$ has been lost.) Although it may be felt here that such a fault is the user's responsibility, we think that the software should not accept bad inputs, but, if it does so, it should process them with some suitable warnings.
- To simplify the inequality $3x > 5x$, the variable domain is to be changed, but the outcome may be quite strange. For example, if the domain is $(-\infty, 0]$, the answer may be "-true".
- CAS may return that $\log 0 = -\infty$. Does this mean that $y = \log x$ is defined for $x = 0$ or that this known singularity has simply not been captured?
- Is $x = \dfrac{-a}{(b-c)}$ the solution to the equation $a + bx = cx$?
- While CAS correctly solves the following equations in R

  $\dfrac{x}{x-3} - \dfrac{3}{x+1} = \dfrac{12}{x^2 - 2x - 3}$ ("false") and $\dfrac{x}{x-2} - \dfrac{3}{x+1} = \dfrac{9}{x^2 - x - 2}$ ("$x = 3$"),

  this may not apply for the equations

  $\dfrac{x^2 - 1}{x - 1} = x + 1$ ("true" or "x"); $\dfrac{1}{x} = \dfrac{1}{x+1}$ ({} or "$x = \pm\infty$" or a "giant" number such as -208334201392506745.23);

  $\dfrac{1}{x+1} = \dfrac{1}{3x-1}, \dfrac{1}{x} = \dfrac{1}{x^2}$ ("$x = \pm\infty \vee x = 1$" or just "$x = 1$"), etc.

3 REQUIREMENTS

Looking ahead we suggest that future CAS (the CAS+) should address the following:

- *Zero* is to be thoughtfully treated by CAS+.
- We know that the majority of students eagerly simplify $\dfrac{x^2 - 1}{x - 1}$ to $x + 1$ without bothering themselves with the requirement $x \neq 1$. Before any simplification of the type "provided that ..., the given can be reduced to ..." CAS+ should require the user to declare the domain of the given underlying function. Otherwise, we will promote the typical paper-and-pencil approach.
- Summarising an Austrian experience of using DERIVE, Heugel (1997) describes how students can solve the task "Determine *a*, *b*, and *c*, so that $4x^2 + a + 25 = (b + c)^2$" by using substitution and factorisation. Instead of

that, the right side of this relation should be written as $b^2 + 2bc + c^2$, leaving CAS to do the rest. To achieve this end, CAS+ may utilise the power of the unification algorithm of the PROLOG language (see the Appendix).

- In the traditional paper-and-pencil environment, typical irrational equations and inequalities are solved by squaring both sides of the given relation. CAS+ should support such a solution process for both equations and inequalities, requiring the user to specify a correct domain of the given relation.
- Solving equations and inequalities in a step-by-step manner should be conceptually supported by CAS+ comments when non-equivalent transformations are applied ("Any solution lost?" and "Spurious Solutions?")

4 CONCLUSIONS

To promote P-C links, CAS+ should therefore: (a) facilitate the procedural work when a concept is being verified; and (b) require the user to think conceptually before a procedure is used. It is true that the problem of verifying equality of different representations is a very complex CAS issue both algorithmically and didactically (see, for example, Kovacs, 1999; Drijvers, 2000; Keng Yap, 2000), but if we want CAS to become an able pedagogical tool that can adequately operate both syntactically and pragmatically (see Ruthven, 2001), some improvements are needed.

REFERENCES

Artigue, M. (2001). Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual Work. Paper presented at CAME 2001, Freudenthal Institute, University of Utrecht. Available: http://ltsn.mathstore.ac.uk/came/events/freudenthal/

Drijvers, P. (2000). Students encountering obstacles using a CAS. *International Journal of Computers for Mathematical Learning*, **5**, pp 189-209.

Haapasalo, L. & Kadijevich, Dj. (2000). Two types of mathematical knowledge and their relation. *Journal für Mathematik-Didaktik*, **21**, **2**, pp 139–157.

Heugel, H. (1997). Experimental and Active Learning with DERIVE. *Zentralblatt für Didaktik der Mathematik,* **29**, **5**, pp 142-148.

Hochfelsner, C. & Kligner, W. (1998). Investigation of the impact of the TI-92 on manual calculating skills and on competence of reasoning and interpreting (in German). Availablet: http://www.acdca.ac.at/projekt2/kl4/test4kt1.htm.

Kadijevich, Dj. & Haapasalo, L. (2001). Linking procedural and conceptual mathematical knowledge through CAL. *Journal of Computer Assisted Learning*, **17**, **2**, pp 156-165.

Keng Yap, C. (2000). *Fundamental Problems of Algorithmic Algebra.* Oxford University Press, Oxford.

Kovacs, A. (1999). Computer algebra: impact and perspectives. *Nieuw Archief voor Wiskunde*, **17**, **1**, pp 29-55.

Ruthven, K. (2001). Theorising the Dialectic between Technique and Concept in Mathematical Activity: Reflections on Recent French Studies of the Educational Use of Computer Algebra Systems. Paper presented at CAME 2001, Freudenthal Institute, University of Utrecht.
Available: http://ltsn.mathstore.ac.uk/came/events/freudenthal/

Zehavi, N. (1997). Diagnostic Learning Activities Using DERIVE. *Journal of Computers in Mathematics and Science Teaching*, **16**, **1**, pp 37-59.

BIOGRAPHICAL NOTES

Dr Djordje Kadijevich obtained his BSc and MSc in mathematics from Belgrade University and a PhD in informatics from the University of Novi Sad (his PhD thesis examines mathematical problem solving through the development of expert system knowledge bases). He works at the Graduate School of Geoeconomics, Megatrend University of Applied Sciences, teaching the courses Introduction to Informatics and Basics of Informatico-mathematical Modelling. Since the academic year 2000/2001 he has taught Didactics of Informatics at the Mathematical Faculty, Belgrade University. He has also worked at the Institute for Educational Research and the Mathematical Institute of the Serbian Academy of Sciences and Arts as a researcher in mathematics and computer science education.

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Further information relating to present research interests and recent publications can be found on his home page at http://www.mi.sanu.ac.yu/~djkadij

Appendix - A simple PROLOG program

```
% prototype #1
% program below recognizes and factors the
difference of squares for typical cases
factor(A-B, (X+Y)*(X-Y)) :-
  find(A, X),
  find(B, Y).
find(A^2, A).
find(A, A^0.5) :-
  atom(A).
find(A^N, A^N1) :-
  integer(N),
  N1 is N/2.
find(A, X) :-
  number(A),
  X is exp(0.5*ln(A)).
```

To enhance C/C++ or other environments with the reasoning power of PROLOG, one may use Amzi! PROLOG (see http://www.amzi.com).

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