where the term - bx_n^2 corresonds to a form of "environmental damping."

At this point, there are two mathematical directions to pursue, both of which are attractive. On the one hand, it is natural to explore the phenomena of overshoot, cycles and chaos that arise when a > 2. On the other hand, it is also natural to explore more realistic rules for change, such as Verhulst's equation in the presence of delays and "noise."

Having used readily available forms of computer technology, students at the pre-calculus level can now be prepared to engage in some meaningful forms of computer modeling. For example, they can anticipate the study of functional differential equations by introducing delays:

$$x_{n+1} - x_n = ax_n - bx_{n-d}^2$$

Using random number generators and the inverse of the normal distribution function, they can anticipate the study of stochastic differential equations by introducing a discrete form of "white noise":

$$x_{n+1} - x_n = ax_n - (b+W(x_n))x_n^2$$

In other words, they are prepared to bring deep mathematical concepts to bear in thinking meaningfully about issues of profound importance to their generation.

These topics have been addressed in several contexts. At the University of California they are part of a summer program for talented students called Cosmos:

http://cosmos.ucdavis.edu/comp_apps.html

Similar ideas were developed in a series of articles in *Quantum* (an American counterpart of the Russian *KVANT*), and they are addressed in a book with G.D. Chakerian, *Iterative Algebra and Dynamic Modeling* (Springer-Verlag, NY, 1999).

I would be very interested in developing contacts with colleagues interested in pursuing similar themes in programs for high school students.

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PROCEDURAL AND CONCEPTUAL MATHEMATICAL KNOWLEDGE: COMPARING MATHEMATICALLY TALENTED WITH OTHER STUDENTS

Djordje Kadijevich, Slavica Maksich, Ioannis Kordonis

Abstract: *Developing both procedural and conceptual knowledge and promoting links between the two are very important goals of mathematics education. This study examines acquiring and linking procedural and conceptual mathematical knowledge by comparing mathematically talented with other students. It reveals that the former students primarily show their strengths in the conceptual domain, where as the latter ones do so in the procedural domain. Having in mind that the learner may prefer (or be bounded to) one learning/thinking style, it is important that the planning and evaluation of an instructional treatment with the two goals in focus should recognize students' learning/thinking styles. Such an approach, involving both mathematically talented and other students, would promote better understanding of their learning and a more adequate interpretation of their mathematical achievements. Some suggestions for educational practice concerning acquiring and linking procedural and conceptual mathematical knowledge are included.*

Keywords: *mathematically talented students, procedural knowledge, conceptual knowledge, learning/thinking styles*

CONCEPTUAL VS. PROCEDURAL KNOWLEDGE

Having in mind this dynamic view of conceptual knowledge, we accept the following proceduralo-conceptual knowledge distinction (Haapasalo & Kadijevich, 2000):

- *Procedural knowledge* denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation form(s). This usually requires not only the knowledge of the objects being utilized, but also the knowledge of format and syntax for the representational system(s) expressing them.
- *Conceptual knowledge* denotes knowledge of and a skilful "drive" along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms.

Procedural knowledge often calls for automated and unconscious steps, whereas conceptual knowledge typically requires conscious thinking. However, the former may also be demonstrated in a reflective mode of thinking when, for example, the student skilfully combines two rules without knowing why they work.

ACQUIRING AND LINKING CONCEPTUAL AND PROCEDURAL KNOWLEDGE

Developing both procedural and conceptual knowledge and promoting links between the two undoubtedly are important goals of mathematics education. However, except for a few studies, research has not explicitly examined means whereby both of these two goals can be achieved (a number of studies dealt with the two knowledge types but not with their links). [1]

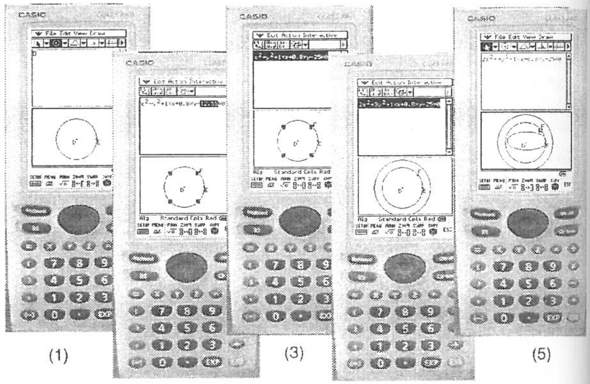
As regards the first goal, it seems that better knowledge gains would be achieved when conceptually-based instruction precedes teaching procedures (Siegler, 2003). This conceptually-favoured approach may be relevant to the second goal concerning computer-assisted learning of mathematics (see Kadijevich & Haapasalo, 2001).

Despite the fact that the philogenetic and ontogenetic nature of mathematical knowledge supports the dependence of conceptual on procedural knowledge (especially in early mathematics education where concepts are comprehended through their usage not definitions), most educators assume that procedural knowledge depends on conceptual knowledge, probably because this position reflects educational needs typically requiring a

[1] A recent ERIC search, which used words "gifted" ("talent" or "talented"), "mathematics", "procedural knowledge" and "conceptual knowledge" revealed no studies on this important topic.

large body of knowledge to be transferred and understood. Learning thus may start by using students' mental models comprising procedural or conceptual ideas[2], which may act like a wake-up voltage in an electric circuit that triggers another, more powerful current to be amplified again. However, no matter which of these learning starting points is chosen (procedural or conceptual), linking the two knowledge types is very likely to occur if they, whenever possible, simultaneously activate each other. The following example, utilizing ClassPad 300 – a powerful calculator made by CASIO[3], illustrates such a simultaneous activation (see Haapasalo & Kadijevich, 2003).

Without knowing anything about the analytic expression of a circle, we can just play harmlessly by drawing a circle in the geometry window (1), and then drag and drop the cirlcle into the algebraic window (2). Something surprising happens: The circle seems to be expressed in algrebraic form $x^2+y^2+0.8xy-12.55=0$. Let's manipulate (3) the equation by changing 12.55 to 25, then drag-and-drop it to see the new circle (4). It seems that only the radius changes. Let's go back to the algebraic window to do more manipulations (5). This time, let's change the coefficients of the second degree variables: $2x^2+9y^2+0.8xy-12.55=0$ seems to make an ellipse.



(1) (2) (3) (4) (5)

It is important to underline that linking the two knowledge types continuosly occurs through the use of symbols whereby mathematical concepts are represented.

"Throughout the learning of arithmetic, algebra and the calculus, the symbolism acts as a pivot to act either as a clue to carry out a procedure (such as adition) or as the output of that procedure (the sum), as in these examples:

symbol	*process*	*concept*
$3+2$	addition	sum
-3	subtract 3, 3 steps left	negative 3
$3/4$	division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\lim_{x\to 2}\frac{x^2-4}{x-2}$, $\sum_{n=1}^{\infty} 1/n^2$	tending to limit	value of limit
$\sigma\in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n

The use of a symbol as a pivot between ***pro***ces and con***cept*** is called a ***procept***" (Tall, 1996; p. 8).

[2] See Vergnaud's (1990) notions of concepts-in-action and theorems-in-action.
[3] See http://www.classpad.org/Classpad/Casio_Classpad_300.htm.

However, despite a high theoretical significance, proceptual learning/teaching has not widely implemented in classroom so far. What kind of learning/teaching seems in general promising for linking the two knowledge types?

As mathematical objects can be seen in several ways - recall here Slavit's (1996) views of function or Thurston's (1998) views of derivative - mathematical objects (concepts, procedures, problems) should be, whenever possible, examined from different perspectives, which strongly suggests that *enabling/utilizing various learning paths* (give several problem solutions, provide different views on the same object/topic, examine different mathematical objects from the same perspective) may be the main feature of traditional and technology-based learning environments that would link the two knowledge types.

Or, to use a "glasses" metaphor, the landscape of mathematics should continuously be viewed with different glasses, the choice and utilization of which should be explicitly taught to students, of course, in a properly-stressed way. Henri Poincaré once said that mathematics is the art of giving the same name to different things, which is a profound idea of the contemporary notion of isomorphism. Sometimes it can also be said, though not in such a profound way, that mathematics is the art of giving different names to the same thing (e.g., 1/2 can stand for a fraction, a rate or simply the division 1 by 2). Good glasses and their proper use seem therefore unavoidable for those wishing to relate procedural and conceptual mathematical issues, and realize what mathematics really is, concerning a general, humanistically-oriented context. Of course, good glasses do not automatically produce good pictures (Kadijevich, 2003).

CONCEPTUAL VS. PROCEDURAL THINKING SUPERIORITY

Feldman (1986) underlines that giftedness is typically expressed within a particular domain where individual demonstrates "performance superior to that of most others" (p. 302). According to him, giftedness can mean various things:

> It can mean *faster* movement through the stages of mastery of a domain; it can mean movement to more *advanced* levels that very few reach; and it can mean *deeper* understanding of each of the levels reached. For a small number of individuals, it may mean all three things (p. 295).

Having in mind the distinction between procedural and conceptual knowledge (or stated simply between skills and understanding; see Byrnes & Wasik, 1991), such distinctions of giftedness may imply that gifted individual demonstrates procedural and conceptual knowledge of a particular domain that are superior to these knowledge types of most others of his/her age. This sort of conclusion is however not empirically supported. For example, Schofield & Ashman (1987) examine cognitive differences between gifted individuals and others concerning successive processing (connectable with procedural knowledge), simultaneous processing (connectable with conceptual knowledge) and planning. They find that, although gifted students did not outperform above average student at successive processing and basic planning, they did demonstrate "the higher level planning/metacognitive and simultaneously processing functions" (p. 18). Does this finding suggest that giftedness should primarily be viewed as conceptual not procedural superiority to the thinking of most individuals?

Although theoretical and empirical evidence is still slight, such a conclusion concerning mathematics education is nevertheless supported by research outcomes summarized below.

- "the *capable*, and especially the *gifted*, have highly connected conceptual structures that enable them to grasp the essential elements in a problem without losing track of the details, to focus on generalities, to curtail reasoning to an essential minimum, and to think flexibly" (Tall, 1996; p. 6).

- Gifted students prefer different ways of thinking. Of 34 gifted students, 6 preferred symbolic and logical reasoning, 5 preferred visually-based thinking, whereas others demonstrated a range of preferences for verbal-logical and visual thinking (Krutetskii cited in Tall, 1996).
- For elementary gifted students, scores on conceptual tasks were higher than scores on computational tasks, probably because the former tasks, as more intuitive and less instructionally-dependent, are more accessible and less error-prone to such students than the latter ones (Lupkowski-Shoplik, Sayler & Assouline, 1994).

ACQUIRING AND LINKING CONCEPTUAL AND PROCEDURAL KNOWLEDGE: A RECONCEPTUALIZATION

It seems that mathematically talented students primarily show their strengths in the conceptual domain, whereas other mathematical students do so in the procedural domain. Furthermore, mathematically talented students may be less interested in procedurally-based teaching (still dominant in our educational practice), whereas their less able classmates may not be interested in conceptual clarifications. What then can we achieve concerning the two goals: developing both procedural and conceptual knowledge and promoting links between the two?

It seems that learner's strategies (cognitive and metacognitive) are influenced and even constrained by some central strategies, such as holist and serialist originated from his/her learning style and approach to learning. According to Entwistle (1988), the serialist strategy is used when the learner is rather concerned with details (usually in the order the material is presented), whereas the holist strategy is employed when the learner is more interested in the presented subjects as a whole, searching for important relations between ideas.[4]

A recent study, which examines scientific thinking (Williams, 2002), makes a distinction between axiomatic and relational thinking styles:

> The axiomatic thinking style "connects information in an axiomatic progression ... it is characteristic of pure mathematics, structured programming, and highly specialized fields. At the other end of the spectrum, the relational thinking style considers more pieces of information simultaneously, and looks for connections following identifies patters ... It is often associated with intuition and is characteristic of applied mathematics, object oriented programming, and interdisciplinary fields" (p. 2).

It underlies that both thinking styles are equally relevant to progress in science.

Despite a simplification, it may thus be said that learning/thinking occurs between the two ends of an agreed learning/thinking style spectrum, and that, no matter which terminology is applied, genuine learning, as progress in science, requires a balance between the utilization of the two styles (recall here proceptual thinking). Such a balance, because of the dominance of one half of the brain and other affective reasons, may be out of reach of many learners (talented or not talented), who, in our terms, may tend to either proceduralize or conceptualize knowledge items. Having realized this, the two goals (developing both procedural and conceptual knowledge and promoting links between the two) and their outcomes should be examined in the context of applied learning/thinking styles. If it turns out that, despite a skilful instructional treatment, some students demonstrate small gains in one knowledge type or have not established the desired links (usually indicated by an insignificant correlation between the scores on the two knowledge

[4] Along with the holist/serialist dichotomy regarding learning, a number of other dichotomies in respect to perceiving (field-dependent vs. field-independent), information processing (relational vs. analytic) and thinking (diverges vs. convergers) have been proposed. See Kadijevich (1993).

items), it nevertheless becomes an expected outcome if the applied measure of students' thinking style evidences their preference for one end of the learning/thinking style spectrum.

To summarize: for both mathematically talented and other students, the planning and evaluation of an instructional treatment with the two goals in focus should recognize students' learning/thinking styles. Such an approach would promote better understanding of their learning and a more adequate interpretation of their educational achievements. Remembering the relevance of the affective domain for problem solving performance (e.g., Schoenfeld, 1992), acquiring and linking procedural and conceptual knowledge may not only be examined in cognitive, but also affective terms. We still know little how cognition and affect interact when learning takes place, but it is sure that gains in procedural and conceptual knowledge and their links, as any cognitive product, are to some extent influenced by the affective domain.

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COGNITIVE VISUALIZATION OF MATHEMATICAL ABSTRACTIONS AS AN EFFECTIVE WAY TO STIMULATE CREATIVITY IN MATHEMATICS.

Alexander Zenkin, Anton Zenkin

Abstract: *The interactive man-machine system 'Pythagoras', based on the Cognitive Computer Graphics (CCG) concept is described. The system produces dynamic, color-musical CCG-images (pythograms) of abstract mathematical objects creating a so-called Cognitive Reality (CR) space. The pythograms practically make visible the semantics of a given problem domain. A student, sinking into the CR-space, gets the unique possibility of interacting directly with mathematical abstractions of a high level by means of visual, color-musical, semantic, and aesthetic, channels simultaneously. Such a CCG-technology allows us to combine the abstract and the concrete, logic and intuition, and left- and right-hemispherical thinking. It is an effective way to make real, non-trivial mathematical discoveries, to generate new mathematical knowledge, and to motivate the teaching process of gifted students in mathematics on the basis of real CCG-discoveries. The last is a very important factor in evoking cognitive enthusiasm, to stimulate analytical, critical logical thinking and to develop the scientific intuition and creative abilities of the gifted students.*

Key words: *Cognitive Computer Graphics, CCG-visualization of Math Abstractions, Activization of intuitive, visual thinking, Unity of Logic and Intuition, Creative Process, Creative Environment, Education via CCG-Discoveries, Creative Motivation.*

INTRODUCTION

One of the most outstanding mathematicians of the XX century, D.Hilbert, wrote: "The infinite! No other question has ever moved so profoundly the spirit of man. [...] the final elucidation of the essence of infinity oversteps the limits of narrow interests of special sciences and, moreover, that became necessary for the honour of the human mind itself."

Indeed, possibly every outstanding mathematician began his/her creative, scientific career with the audacious attempt to comprehend the essence of infinity. Such the attempts are usually peculiar to young people and assist them to develop intuitive, creative thinking. Is it possible to aid them? Raymond Smullyan, in his recent remarkable book "Satan, Cantor, and Infinity", remarks apropos of this: "... it must be wondered at that the whole fascinating subject of Infinity is so little known to the general public! Why isn't it taught in high schools? It is no harder to understand than algebra or geometry, and it is so rewarding! [. . .] Even a neophyte can understand [...] an account of what may be the greatest mathematical problem of all time which remains unsolved to this very day!"

The problem of infinity is closely connected with the main notions of science, starting from ancient times: 'discrete' and 'continuous'. The notions are a conceptual basis of mathematics, physics, biology, philosophy, etc. The most adequate mathematical model of 'discrete' is the *infinite* series of *finite* natural numbers:

$$1, 2, 3, \ldots \qquad (*)$$

The most adequate mathematical model of 'continuous' is the set of all points or real numbers of the segment [0,1].

Cognitive visualization of these main objects of mathematics and their main mathematical properties makes them much more accessible for young people and aids to develop their creative possibilities.