Linking procedural and conceptual knowledge

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After summarizing possible distinctions and relationships between procedural and conceptual knowledge, this study examines possible means whereby these two knowledge types can be related. Through considering suitable tasks assessing procedural and conceptual knowledge and their links, evolved is the main feature of traditional and technology-supported learning environments that would promote the desired links. Suggestions for further research are included.

Terminological clarification

The dinstinction between procedural and procedural mathematical knowledge seems appropriate at both theoretical and practical levels. Although such a distinction has been defined/described in various ways, most of them can be included in the following categorization made by Haapasalo & Kadijevich (2000, 141).

- *Procedural knowledge* denotes dynamic and successful utilisation of particular rules, algorithms or procedures within relevant representation form(s), which usually require(s) not only knowledge of the objects being utilised, but also knowledge of the format and syntax for the representational system(s) expressing them.
- *Conceptual knowledge* denotes knowledge of and a skilful 'drive' along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms.

While procedural knowledge often calls for automated and unconscious steps, conceptual knowledge typically requires conscious thinking. However, the former may also be demonstrated in a reflective mode of thinking when, for example, the learner skilfully combines two rules without knowing why they work.

Does procedural rely on conceptual knowledge or vice versa?

According to Shimizu (1996, 234), "*understanding how procedural knowledge and conceptual knowledge relate to one another is one of the major foci in mathematics education*". The literature evidences that researchers/educators basically assume the reliance of conceptual knowledge on procedural knowledge or vice versa, which, respectively, implies the following educational strategy: "*use procedural knowledge and reflect on the outcome*" or "*build meaning for procedural knowledge before mastering it*". The former position, called *developmental approach*, reflects the developmental nature of mathematics, especially in early mathematics education, whereas the latter, called *educational approach*, seems to fulfill educational needs typically requiring a large body of knowledge to be understood. (Haapasalo & Kadijevich 2000)

While the developmental approach may be suitable for introducing the concept of a limit that promotes its dynamic definition, the educational approach may be appropriate for teaching fractions and decimals. According to Vygotsky (1978), procedural knowledge does precede conceptual knowledge ontogenetically, but it is school learning that frequently precedes intellectual development. Such a position suggests that, for most topics, the educational approach may be more relevant than the developmental one. However, the utilization of an interplay between these approaches may, for some topics, be a better strategy than the application of just one of them. This interplay is highlighted by Haapasalo (2003, xx-yy), stressing the importance of pedagogical theories guiding the learning process.

Are links between procedural and conceptual knowledge promoted in practice?

There is no doubt that mathematics education should, among other important goals, develop both procedural and conceptual knowledge and make links between the two. However, a small number of studies examined the effects of their treatments on the coordination of two knowledge types. As regards the traditional mathematics education, Nesher (1986) and Shimizu (1996) evidence no links between procedural and conceptual knowledge (hereafter ***P-C*** links). As regards a computer-based mathematics education, the reported outcomes are contradictory. While Yerushalmy (1991), Hochfelsner & Kligner (1998) and Laborde (2000) report the lack of ***P-C*** links, Schwarz *et al*. (1990), Simmons & Cope (1997) and Kadijevich & Haapasalo (2001) evidence that ***P-C*** links can be established. A summary of the favourable findings is given below.

- Having utilized a computer-based, three-fold function representation dealing with algebraic, graphic and tabular forms, at least 43% of the subjects established ***P-C*** links (Schwarz *et al*. 1990).
- A restricted LOGO feedback, displaying the screen turtle's position only, could make ***P-C*** links concerning angle and rotation (Simmons & Cope 1997).
- ***P-C*** links were promoted through the development (and refinement) of expert system knowledge bases comprising various facts and rules. Such links were also established through the construction of conceptual knowledge based upon a five-step framework involving verbal, graphic and symbolic concept representations (Kadijevich & Haapasalo 2001).

In what way can P-C links be established?

Although research has not explicitly dealt with this question, initial answers can be extrapolated from the literature. Four of them are summarized below. It may be said that while the first two views reflect the developmental approach, the rest ones assume the educational approach.

- *Use multiple representations coordination* — Learner usually divides the world (not a problem) into several (conceptual) *microworlds*, enabling different procedures to be applied within each of them (e.g. adding numbers in little worlds based upon finger manipulation, money facts and LOGO turtle geometry facts). It is basically the elaboration and coordination of these microworlds that enables conceptual knowledge to develop out of such fractured procedural knowledge. (Papert 1987)
- *Foster proceptual thinking* — A *procept* is "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either of both". ***P-C*** links are promoted through *proceptual thinking* by utilizing "procedures where appropriate and symbols as manipulable objects where appropriate". (Gray & Tall 1993, 6-8)
- *Exercise production rules utilization* — New task-specific productions (condition-action rules) have been initially developed through applying the available conceptual knowledge interpretively by means of some general problem-solving productions. These newly generated skills comprise procedural knowledge after collapsing of a sequence of productions into a single one whose utilization doesn't require conceptual knowledge retrieval. It is therefore production rules utilization that enables ***P-C*** links. (Anderson 1983)

- *Apply competence items utilization* (educational approach) — It is *utilization competence* comprising various *enabling conditions* that makes ***P-C*** links possible. In geometric tasks calling for locus constructions (e.g. construct a triangle, being given $a + b$, c, and b), an enabling condition (an utilization competence item) is typically the following rule: "To determine a point that lies on a line with certain properties, construct this line obtaining a locus for that point". (Gelman & Meck 1986, 30)

While Schwarz *et al*. (1990) and Haapasalo's five-step framework (Kadijevich & Haapasalo 2001; more detail in Haapasalo 2003) utilize multiple representation coordination, Kadijevich's empirical research (Kadijevich & Haapasalo 2001) applies production rules utilization.

Implications for theory and practice

Probably because of its high complexity, the question of ***P-C*** links has been only partially (and mostly implicitly) addressed so far. Having in mind their strong educational relevance, ***P-C*** links need to be examined in more detail both on theoretical and empirical levels. Three directions seem particularly relevant: (1) clarify the nature of ***P-C*** links, (2) design traditional and technology-supported learning environments promoting ***P-C*** links; and (3) develop tasks, assessing the two knowledge types and their links. Although these directions are highly interrelated — (1) calls for (3), (2) is based on (3) both influenced by (1), etc. — the last direction is probably the most important since, without a proper measure, the two knowledge types and their links cannot be adequately addressed, which then cannot yield improvements in directions (1) and (2). Despite the fact that tasks/problems may be (very) person, content and context sensitive (especially if a constructivist position is assumed), the rest of this section summarizes some research findings relevant to directions (1) - (3) that may advance our knowledge on ***P-C*** links.

- *Direction (3)* — By examining procedural and conceptual demands embedded into tasks on modelling, Galbraith & Haines (2001), distinguish three task types: (1) *mechanical* involving routine calculations (e.g., factorize $x^2 – ax + 12$ for a =5, 6, 7 and 8); (2) *interpretative* requiring conceptual conclusions (e.g., determine the position of vertex for $y=x^2 – ax + 6$ as given by a); and (3) *constructive* involving links between procedural and conceptual knowledge (e.g., given graphically 1/sin$x$ and 1/cos$x$, draw y = 2/sin$x$ + 2/cos$x$). A promising approach is to develop such groups of tasks, and, of course, to verify their categorization psychometrically. (Can we confidently speak about a classification/ taxonomy and utilize it trustworthily if it has not been (cannot be) reflected in the subjects' scores on assessment items?) This requirement is particularly relevant to technology-based mathematics education, where more time can be devoted to conceptual matters. An example of the three task types taken from Leinbach et al. (2002) is given below.

> Using the random number generator in your CAS generate a general cubic function of the form $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are random values $>< 0$. Adjust the range of your graphing window so that you have a reasonable view of the graph of $f(x)$.
> *Mechanical* - Locate the centre of symmetry for the graph of $f(x)$. Find a transformation to make this centre of symmetry lie on the x-axis. Graph the transformed function.
> *Interpretative* - Using the substitution operator in your CAS, replace x by $(x - s)$ for $s = \pm 1, \pm 2$. Graph these new functions, describe the result of this substitution.
> *Constructive* - Consider a general cubic function $f(x) = ax^3 + bx^2 + cx + d$, find a shift of the graph of f that will eliminate the squared term in the expression for the shifted function. (p. 5)

A credible rationale for such a categorization[#] can be found in Slavit's (1996) three views of functions related to action-oriented objects (function as a computational machine), object-oriented objects (function as a graph describing how independent and dependent variables change) and property-oriented objects (function as an object having local and global properties).

- *Direction (2)* — A crucial point in Slavit's examination of functions is not that there are exactly tree views of functions but that functions can be viewed in several ways. The same applies for other mathematical objects. A quote from Thurston (1998) is relevant here.

> There is a number of ways of "*thinking about* or *conceiving of* the derivative, rather than a list of different *logical definitions*. ... Unless great efforts are made to maintain the tone and flavour of the original human insights, the differences start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions. (p. 341)

[#] No matter how a task classification/taxonomy is theoretically and/or empirically grounded, one can always debate its description and operationalization since, as underlined, task/problem solving may be (very) person, content and context sensitive. However, to guide and foster adequate mathematics learning as well as achieve a comprehensive evaluation of its outcome, mathematical competencies (knowledge and skills) should be viewed, taught and assessed by means of a suitable task classification/taxonomy used as a useful framework not as a dogmatic recipe.

Since a mathematical object can be a solution process (recall that a solved problem may introduce a new concept or rule), problems should also be solved in several ways.[&] This means that mathematical objects (concepts, procedures, problems, etc.) should be, whenever possible, examined from different perspectives, which strongly suggests that *enabling/ utilizing various learning paths* (give several problem solutions, provide different views on the same object/topic, examine different mathematical objects from the same perspective, etc.) may be the main feature of traditional and technology-based learning environments that would make ***P-C*** links possible. Or, to use a "glasses" metaphor, the landscape of mathematics should continuously be viewed with different glasses, the choice and utilization of which should be explicitly taught to students, of course, in a properly-stressed way. Henri Poincaré once said that mathematics is the art of giving the same name to different things, which is a profound idea of the contemporary notion of isomorphism. Sometimes it can also be said, though not in such a profound way, that mathematics is the art of giving different names to the same thing (e.g., 1/2 can stand for a fraction, a rate or simply the division 1 by 2). Good glasses and their proper use seem therefore unavoidable for those wishing to relate procedural and conceptual mathematical issues, and realize what mathematics really is, concerning a general, humanist-oriented context. Of course, good glasses do not automatically produce good pictures. To avoid a data overflow in learning, it is often appropriate to simplify the learning situation as described by Haapasalo (2003, xx-yy).

- *Direction (1)* — "The relation between computational expertise and conceptual understanding, and how each supports the other, is complex and requires careful study and thought." (Howe, 1998, 244). Such a study may refine multiple representations coordination that seems appropriate not only for the developmental approach but also for the educational one (see Haapasalo 2003). Remembering the relevance of the affective domain for problem solving performance (e.g., Schoenfeld 1992), the nature of ***P-C*** links may not only be examined in cognitive, but also affective terms. We still little now how cognition and affect interact when learning is taking place, but it is sure that ***P-C*** links, as any cognitive product, are to some extent influenced by the affective domain.

[&] According to the NCES document *Pursuing Excellence*, relating to the Third International Mathematics and Science Study (NCES 1996), this activity was a distinctive feature of Japanese mathematical teaching, which enabled Japanese students to obtain considerably higher test scores than U.S. and German students who were taught in the traditional way emphasizing skills rather than understanding.

Coda

Computers can be used as cognitive tools, and having in mind that multimedia may be a powerful tool for knowledge construction (Jonassen 2000), future mathematics teachers should design multimedia lessons enabling/utilizing various learning paths, which would help them and their students establish ***P-C*** links. Although the explicit treatment of ***P-C*** links is a very complex enterprise for multimedia designers, successful multimedia lessons can still be developed (Kadijevich 2002).

"*A very important research question regarding computer-based mathematics education is how different technologies affect the relation between procedural and conceptual knowledge*" (Kaput 1992, 549). Although this research direction has been put forward some ten years ago, very little has been done to uncover in what ways, to what extent and for what kind of students ***P-C*** links can be promoted in different computer-based environments, such as spreadsheets, dynamic geometry software, CAS and multimedia software. Further research may primarily address these questions promoting more appropriate learning activities.

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