Linking Arithmetic to Algebra

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A key issue of early algebra education is a selection of topics which may link arithmetic to algebra. This paper considers a number of such topics: first steps in addition and subtraction, invariant manner of expressing arithmetic rules procedurally, equations, inequalities and problems of discovering rules.

Research on school algebra has evidenced how an inadequate arithmetic experience may limit the learning of algebra (see, for example, Kieran, 1992). Although a recent ICMI discussion document underlines that the relationship between arithmetic and algebra is an ongoing concern of early algebra education (ICMI, 2000), it does not particularly address a selection of topics which may link arithmetic to algebra. Our selection of such topics is based upon our involvement in the design of mathematical curricula in Serbia and the pre-service and in-service mathematical education of elementary-school teachers, as well as our experience as authors of some mathematical textbooks for elementary schools. Such a ground enables us to say which links can be elaborated with ease and understanding.

The question of which topics to select and why to do it requires to retrospect the historical development of mathematics and to undertake a thorough subject analysis. That usefulness in life is a valid criterion for selection of and emphasis on subject matter goes without saying, but our understanding also includes all the years a student spends in school.

We have selected to present here those topics which should be elaborated during the first four or five years of teaching mathematics and which are traces of algebra intentionally involved in the flow of the arithmetic course.

First steps in addition and subtraction

Already at the elementary level, when working with numbers in the set $\{1, ..., 10\}$ (and/or reacting to some situations in a natural environment or to their pictorial representations that call for addition and subtraction), children are taught to use plus and minus signs to compose simple arithmetic expressions. They also use the equality sign when a composed expression is equated to a number. Here, and in a specific form, we see at work a little piece of the algebra apparatus.

As it is easier to count on from 7 to 9 than from 2 to 9, it is easier to find the sum $7 + 2$ than $2 + 7$. Many children spontaneously discover this rule of interchange of summands and many teachers state it rhetorically.

Another important rule is involved in the method of addition when the "10-line" is crossed. For example, the sum $7 + 6$ is found in the following way:

$$7 + 6 = 7 + (3 + 3) = (7 + 3) + 3 = 10 + 3 = 13,$$

and here we see at work the rule of association of summands. Such a decomposition into steps, leads to a thorough understanding of this method and, thinking of the links under consideration, is a motivated instance of expressing procedurally the rule of association of summands. Note that the method of subtracting down over ten is another example of a motivated use of arithmetic expressions.

Without any use of letters, these activities seem as being not immediately linked to algebra. But here a formed skill of operating on arithmetical expressions transfers easily to algebra and links directly to the numerical evaluation of literal expressions.

Invariant manner of expressing arithmetic rules procedurally

When we speak our mother tongue, we may do it very correctly without knowing to express in words grammatical rules which govern its regularity. In the same way, children can perform arithmetic operations correctly, example by example, without knowing to explain rhetorically or symbolically the applied arithmetic rules. In such a case, we speak of a procedural manner of expressing things and this manner is not only the first one to be learnt but also the most important form of knowledge to be acquired at early stages of arithmetic teaching.

When we associate summands we may, for example, produce the equality

$$8 + 6 = 10 + 4$$

or

$$8 + (2 + 4) = (8 + 2) + 4.$$

In the former equality we do not express the underlying rule of association invariantly, whereas in the latter one such an expression is present. This means that, in general, only the letter equality keeps to hold true when some of its number(s) is (are) replaced by another one(s).

A more illustrative case is that of expressing the rule of multiplication of sums by a number. (Instead of a technical, we use here a didactical term.) While the equation

$$3 \cdot (5 + 2) = 15 + 6$$

does not have the invariant form, the equation

$$3 \cdot (5 + 2) = 3 \cdot 5 + 3 \cdot 2$$

does have it.

How are children induced to express arithmetic rules invariantly? Requirements like "Write the missing numbers so to get what is true: $4 + 5 = \_\_$, $\_\_ + 3 = 8$, $8 - \_\_ = 5$, etc." are easily understandable forms of programmed exercises even for first graders. A similar use of place holders keeps this manner of expressing being invariant. For example,

$$7 \cdot (5 + 2) = 7 \cdot \_\_ + \_\_ \cdot 2, \quad 7 \cdot (7 + 1) = \_\_ \cdot 7 + 7 \cdot \_\_ , \quad \text{etc.}$$

Why is the invariant manner so important?

When children have formed the feeling that in such relations an involved group of numbers can be replaced by any other, later (when they are third graders) they express easily such rules in a symbolic manner, by replacing specific numbers by letters. Of course, this ability supposes that children have been trained to use letters in the role of variable. Thus, fundamental rules of algebra are built upon arithmetic practice without which they would be a mere play with letters i.e. rules without reason.

The usage of place holders is widely and successfully spread in the Serbian elementary schools to promote an active way of learning. These holders may also be misused as it is so well illustrated in Freudenthal (1978).

Equations - their role and how to solve them

In many countries, equations can be seen included in the first grade arithmetic curricula. And in many cases they can be seen introduced abruptly in a form of groups of simple equalities containing "x" (or another letter) and just given to be solved. Against such a practice, this first explicit step into algebra deserves a careful elaboration. For example, neither "x" nor any combination of signs such as "$x = 3$", "$x + 4$", "$x + 4 = 7$ " have an *a priori* meaning and each of them should be treated separately. Without entering into such details, mention that, in this context, "x" plays the role of a label for a fixed but hidden number. (Technically, "x" is an unknown.) Going from example to example, "x" takes different values and so, slowly, becomes a variable in the set of natural numbers, and this is surely the most important aspect of this teaching theme in its early stage.

How to solve equations?

First, of course, "x" is to be "viewed" as a place holder, and an equation like $17 - x = 8$ is solved by converting it into a suitable question such as "Which number taken from 17 leaves 8?". However, this way of solving equations is rather limited to a set of small numbers (up to 20). Another way that deserves to be called a method is based on the interdependence of arithmetic operations.

Many textbooks underline that addition and subtraction are two operations opposite to each other. It is probably better to say that these two operations are interdependent. When expressed precisely, this interdependence means that whenever one of the following three equalities

$$a + b = c, \quad c - a = b, \quad c - b = a$$

is true or false, the same applies for the other two. In a didactically transposed form, this means that, followed by a variation of numerical data and requirements, all three equalities should be attached to the same situation which bears a visual meaning. Seeing the symmetric property of equality relation mainly as a matter related to the way how a piece of syntactic apparatus is used, children should be systematically trained to associate with such situations all eight equalities:

$$a + b = c, \quad b + a = c, \quad c - a = b, \quad c - b = a, \quad c = a + b, \quad c = b + a, \quad b = c - a, \quad a = c - b.$$

Then, children rewrite $17 - x = 8$ as $x = 17 - 8$, and easily solve this and similar types of equations, which combine the signs of the four arithmetic operations. (Some types of more complex equations in which "x" appears just once could also be admissible. For example, $3 \cdot (2x - 4) = 18$, solved first for $2x - 4$.) To such types of equations some still interesting word problems can be associated. A practice of teaching four graders to solve more complex types of equations, and to achieve this by using formal rules, is nothing more than a bad misconception of some authors of textbooks and some too ambitious teachers.

Some thirty years ago, when equations were introduced in the Serbian first grade mathematics curriculum, majority of the teachers were inclined to banish them from their teaching practice Today, majority of them are defenders of such a teaching matter considering it easy and attractive for children.

Inequalities - their role and how to solve them

Today, in many countries, inequalities are also a topic of the mathematics curriculum designed for the arithmetic course. When, in some cases, the demand to consider only simple inequalities is stated, it does not sound so informative. When, in other cases, the types of simple inequalities are specified like: $x \pm m \geq n, ..., x : m \leq n$, we are not ready to take them all as simple. For instance, all numbers divisible by 7 from 42 on are solutions of the inequality $x : 7 > 5$. And divisibility is a theme usually assigned to fifth graders. On the other hand, when, in some textbooks, inequalities are treated only as problems given to be solved, then we are left to guess which way it should be done (excluding, normally, more formal procedures).

Contrary to equations, inequalities are rarely connected with some interesting word problems. On that ground, many teachers consider them unrelated to life and so a needless topic. As a matter of fact, we find the question of right place and right way of treating inequalities still open. Having in mind the current practice in the Serbian elementary schools, we cannot report anything as a model. To form a statistical average of teaching effects going from classes where this topic is almost completely ignored to those where it is overdone would be a nonsense. Instead of it, we shall shortly summarize our opinion of why, what and how we may teach inequalities at this early stage.

Since "x" figuring in an inequality is already in the role of a variable, such a use of letters is on the best way of linking arithmetic to algebra. Simple inequalities of the form: $x \pm m > n, \; m \pm x \leq n, ...$ can easily be solved by third graders. In case of small numbers, an instructive "method" of solving them is a tabular listing of values of x and those of the left hand side expressions, when x runs from number to number. In case of bigger numbers, the method is based on the following important rules: the sum increases when a summand does so, the difference increases when the minuend does so, the difference decreases when the subtrahend increases and so on, including their variations obtained by the interchange of "increases" with "decreases". For example, the inequality: $179 - x < 54$ is solved by solving the equation $179 - x = 54$, first. Its solution is 125, so the set of solutions of the given inequality is $\{126, 127, ..., 179\}$.

As seen, even in this simple case of solving inequalities in the set N, a logical combination of the conditions $x > 125$ and $x \leq 179$ is met. Thus, inequality solving combined with divisibility could be a perfect ground to start with establishing a precise meaning of conjunctive words "and", "or", "if ... then ..." in a course for, say, fifth graders. In conformity with it, teachers in elementary schools and the authors of mathematical textbooks for such schools should be acquainted with the rudiments of mathematical logic. To achieve this end, a chapter from Kolmogorov (1988) may be a good inspiration. Note that our teaching of such digested rudiments at Teacher's Training Faculty, University of Belgrade has turned in a very encouraging experience.

Discovering a rule

Given a sequence, say,

$$4, 7, 10, 13, \ldots$$

a typical quiz requirement is to discover a rule of its formation, which already manifests itself in case of the given terms and can be applied to find the following one. Observing a regularity that each given term is for 3 greater than the preceding one, we have discovered a simple rule according to which the fifth term of this sequence is equal to 16. However, by using this rule, we cannot easily find the term at the 177th place. To do it, we have to discover a rule connecting the rank of a place with the number at that place i.e. the general term. This leads us to consider the given sequence as the following correspondence:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 4 & 7 & 10 & 13 & \ldots \end{array}$$

and to discover an expression depending on n, writing it where the holder indicates its place:

$$\begin{array}{c} n \\ \downarrow \\ \_\_\_ \end{array} .$$

(By solving an easier problem for the sequence 3, 6, 9, 12, ...), the expression we are searching for is $3n + 1$. In particular, the term at the 177th place is 532.

When rules are limited to the expressions of the form $an \pm b$, many fourth graders find this type of problems interesting and easy. At the same time, they are searching for a correspondence spontaneously using free variable "n".

One of us, working with a group of children supposed to have a strong interest in mathematics, derived first the following equalities pictorially:

$$(1) \quad \frac{1}{2} = 1 - \frac{1}{2}$$

$$(2) \quad \frac{1}{2} + \frac{1}{2^2} = 1 - \frac{1}{2^2}$$

$$(3) \quad \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 1 - \frac{1}{2^3}$$

Then, given was the assignment to find the equality at the nth place

$$(n) \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

(Of course, there are several subtleties, not being the matter of usual concern, which have to be covered before such an assignment is given.) Three out of six third graders and eight out of eleven fourth graders wrote correctly

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n} .$$

This demonstrates that children may use letters easier than majority of us suppose it to be.

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The problem of linking arithmetic to algebra has to undergo a detailed subject analysis followed by carefully designed experimentation. Only then we shall be thinking of it in a well founded way.

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