SOME ASPECTS OF VISUALIZING GEOMETRIC KNOWLEDGE: POSSIBILITIES, FINDINGS, FURTHER RESEARCH *

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Having underlined a rationale for visualizing the knowledge of geometry, the paper describes two tools whereby computer-assisted visualization can be realized. It then summarizes research findings regarding the use of these tools in mathematics education, suggesting promising directions for further research.

1. To visualize geometric knowledge or not?

Whenever possible, teachers/learners should do it. Appropriate grounds for this approach can be found in philosophy, psychology and mathematics education. A summary of such grounds is given below.

- Recalling a medieval distinction between *ratio* (producing an appropriate chain of reasoning) and *intelligentia* (comprehending a totality at once), it is useful to differentiate the knowledge of a mathematical truth from a holistic understanding of the same truth, (see [2]).
- It has been generally acknowledged that students advance their geometric thinking by climbing over the following levels: visualization, analysis, informal deduction, formal deduction and rigor, some of the latter ones may be unreachable "peaks" for many students (see [11]).

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- "The National Assessment of Educational Progress found in 1982 that proof was the least liked mathematical topic of 17-year-olds, and less than 50% of them rated the topic as important" (see [1, §. VII]).

As it may appear at first sight, visualizations are not limited to argumentation based upon pictures (such "proofs" can be found in [8], for example). Instead, they in general deal with drawings that may aid/advance our understanding of various geometric entities and processes. Such visualizations can be made alive by the use of computers – imagine that the square above the hypotenuse in Fig. 1. has been transformed into that pentagonally-shaped shaped object, by dragging point B along the drawn line perpendicular to the hypotenuse.

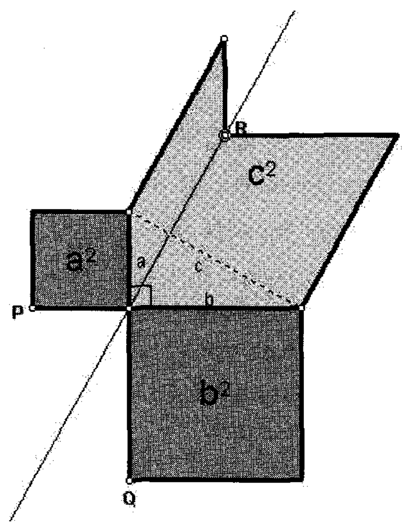

Fig. 1. Is $a^2 + b^2$ equal to c^2 ? (screenshot concerning a demo Geometer's sketch)

2. Two tools for computer-based visualizations

2.1. Dynamic geometry environments

These environments, such as Cabri-geometry (http://education.ti.com/us/product/software/cabri/features/features.html), the Geometer's Sketchpad (http://www.keypress.com/sketchpad/index.html), and Euclid (http://www.dynageo.de/), simulate the Euclidean ruler and compass constructions, support such constructions by utilizing user-defined macros, and allow moving certain parts of a constructed figure without changing its underlying geometric relations. By using such an environment, students can make and test their own conjectures. Two screenshots concerning a simple use of Geometer's Sketchpad are presented in Fig. 2. Note that sketches from this software can be published on the Internet as Java applets by means of the JavaSketchpad program.

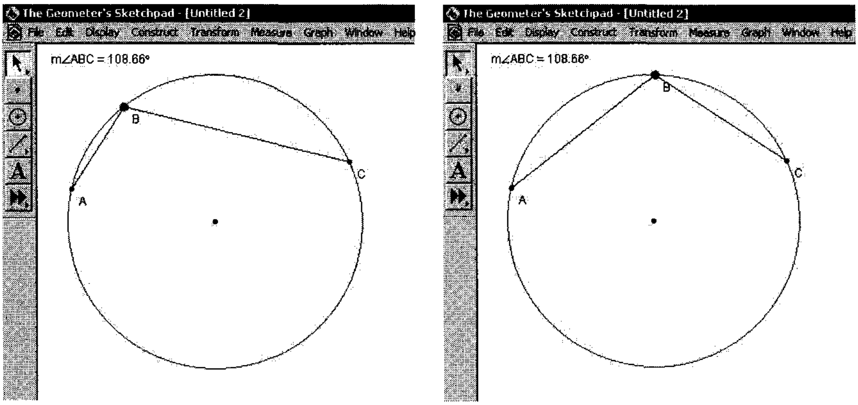

Fig. 2. Would $\angle ABC$ change if point B is dragged along the circumference of the circle?

2.2. Java applets

These applets are programs written in Java http://java.sun.com, which are usually run inside a Web browser such as Microsoft Internet Explorer. Such programs support the creation of (dynamic) pictures, the elements of which can be:

- displayed in a step-by-step fashion, which is applicable to describing a construction task in geometry, to generating the sine curve by using the unit circle, or to providing a hint in problem solving;
- transposed and rearranged, which is suitable for a visual "proof" of Pythagorean theorem;
- dragged, which is beneficial to detecting geometric invariants, like the centroid of a triangle;
- defined by some input data, which is applicable to graphing a function given by equation, to selecting a Platonic solid and an angle of its view, or to selecting the F distribution with the specified degrees of freedom;

- matched against a specified rule, which is suitable for connecting algebraic and graphical representations of functions; and so on.

Skillfully developed collections of Java applets regarding mathematics education can be found at http://www.ies.co.jp/math/java/ (Manipula maths applets), http://www.saab.org/moe/start.html (Maths online), and http://illuminations.nctm.org/pages/912.html (Illuminations of NCTM). Fig. 3. presents a screenshot of an applet taken from the Manipula site.

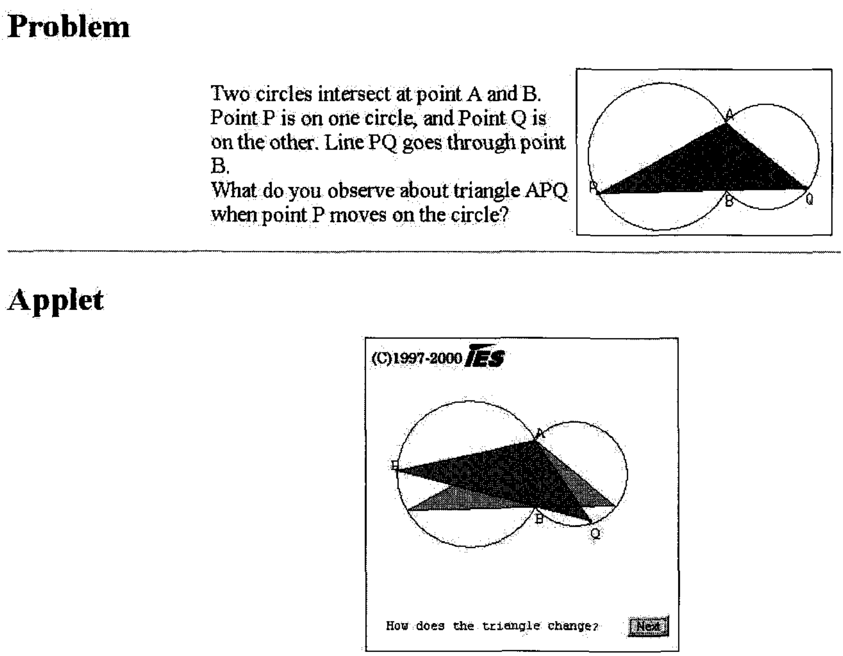

Fig. 3. Task on triangle similarity

It is true that some visualizations can be produced with a pencil much easier than with the presented tools. But, the dynamic aspects of modifying a drawing can only be realized with such tools. Thus, the emphasis concerning visualizations should be on proofs based upon simple animations, which would, in a longer run, change the attitude of proof-denying students.

3. Research outcomes and further directions

A summary of main research findings and suitable directions for further research is given below.

- Although some features of appropriate geometric visualizations can be found in the literature (see, for example, [2]), a detailed list of these features, based upon cognitive, didactical or other issues, still awaits to be developed.
- Problem solving in school geometry requires moves between a theoretical (T) domain, comprising theoretical geometric entities and their properties, and a spatial-graphical (SG) domain comprising spatial-graphical geometric entities and their properties. A key point to success in problem solving is thus linking these domains appropriately. An experience with Carbi geometry evidences that such a linkage is not fully established [7]. Such an unfavorable outcome undoubtedly calls for developing DGE-based instructional units that can promote the links in question (T-SG links), which has not been so far explicitly treated in developed learning/teaching materials (see, for example, [1]).
- To develop T-SG appropriate DGE-supported lessons, the designer would not only utilize a problem-solving context where empiricism and deduction coexist and reinforce each other, but also help students realize the necessity of deductive arguments. Otherwise, empiricism will prevail, resulting in the following unfavorable outcome:

  > Because it would be easy for students to check conjectures, they might develop a **modus operandi** that consisted of making a guess and trying it on a large number of cases. If the guess worked for all those, most of the students would feel no need for proof at all (see [10, p. 262])

  The designer should also be aware that the drag mode is not heuristically neutral, since points that move and line segments that stretch and reduce differ from the traditional, paper-and-pencil entities, which promotes new styles of reasoning (see [3]).
- Although mathematics educators have realized promising values of the use of Java applets-based courses (see, for example, http://www.geometria.de/; see also Java View at http://www.javaview.de), recent electronic searches of the ERIC and MATHDI databases at http://www.askeric.org/Eric/adv_search.shtm and http://www.emis.de/MATH/DI/en/quick.html, respectively, evidenced no studies concerning cognitive and affective features of

Java applets-based learning of mathematics. Such studies, like that of [9], may primarily be published in an on-line journal such as ON-Math (see http://my.nctm.org/eresources/journal_home.asp?journal_id=6).

- Having in mind that multimedia may be a powerful tool for knowledge construction (see [4]), future mathematics teachers should design Java applets-based multimedia lessons enabling/utilizing various learning paths, which would, inter alia, help them and their students establish links between procedural and conceptual mathematical knowledge (see [6]). Although the explicit treatment of these links (or T-SG links in particular) is a very complex enterprise for multimedia designers, successful multimedia lessons can still be developed [5].

4. Coda

Kant once said: "Perception without conception is blind; conception without perception is empty." The message is clear: empiricism and theory are equated parts of any visualization, one influencing the other. Although computer-based visualizations still add a new dimension to learning activities, expectations regarding their utility should be realistic and not just technology-minded since, as Begle already underlined, "mathematics education is much more complicated than you expected even though you expected it to be more complicated than you expected."

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