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Promoting the Human Face of Geometry in Mathematical Teaching at the Upper Secondary Level

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The article examines four activities whereby the human face of geometry may be promoted in mathematical teaching at the upper secondary level. The activities deal with the historical, epistemological, structural and applicative issues of geometric knowledge.

INTRODUCTION

> We are being continually fed with definitions and proofs, presented as though these were revealed truths, but the reasons for these were never given and most of the proofs seemed to spring from the teacher' s chalk as though by magic. —Jean-Louis Laurière

Although the knowledge of mathematics has mostly been acquired by heuristic reasoning and brushed up by deductive reasoning, geometry lessons at the upper secondary level do insist on presenting only the deductive side of the mathematical coin, which may, and frequently does, create a false view of what geometry is and how its knowledge has been developed. This problem of mathematics in general and geometry in particular has been realized in many countries, some of which try to rebalance mathematics curricula by putting less emphasis on the deductive features of mathematics (Quadling 1985). In the United States and Canada, for example, mathematics curricula have been developed under the influence of a recent NCTM document entitled "Curriculum and evaluation standards for school mathematics" (NCTM 1989). This document underlines the relevance of the concept of proof only to post-secondary education, and suggests two-column proofs and the axiomatic system of Euclidean geometry as topics to be de-emphasized in the geometric curriculum. As regards Denmark, mathematical teaching at the upper secondary level is now explicitly required to include the aspect of history, the aspect of modeling, and the internal structure of mathematics, in its overall treatment of mathematical topics (Kadijevich 1993). Having in mind this Danish approach, it is reasonable to require that mathematical teaching primarily displays a human face of mathematics rather than a platonic, or algorithmic, or formalistic, or computerized, or a mechanical face of the subject (Davis 1992). This is especially true for geometry teaching at the upper secondary level since “logic no more explains how we think than grammar explains how we speak; both can tell us whether our sentences are properly formed, but they cannot tell us which sentences to make.” (Minsky 1988, p. 186)

Does mathematics exhibit a human face?

From the time of Giovani Battista Vico (1668–1744), or even earlier, the sciences and the humanities have been considered as two distinctive provinces of human endeavour. According to Davis (1993), mathematics displays features of both science and humanity, which enables it to bridge the gap between the scientific and the humanistic cultures. However, as he underlines, mathematical educators largely ignore not only the human values of mathematics, but also the points of interaction between mathematics and the humanities, which results in hiding this human face of mathematics from the wider public. It is therefore important to uncover the existence of the human face of mathematics (e.g., Blaire, et al. (1992), White (1993) and Ernest (1994)). Furthermore, we need to develop teaching and learning activities that promote this face of the subject.

This article examines four activities whereby the human face of geometry may be promoted in mathematical teaching at the upper secondary level. These are;

- examining wrong and inadequate items from the philogenesis of knowledge,
- demonstrating the ways of creating and testing items of knowledge,
- considering proving as a form of social interaction, and
- examining the use of items of knowledge in modeling the reality.

These activities are examined in the following sections by using various items of geometric knowledge. Although some of these items may be relevant to the lower secondary level, the examined activities, especially the first three, are primarily intended for mathematical teaching at the upper secondary level.

ACTIVITY 1 — Examining wrong and inadequate items from the philogenesis of knowledge

We are almost always condemned to experience errors in order to arrive at truth.
—Denis Diderot

Activity 1 is based upon examining wrong and inadequate items of knowledge that have been created and used in the course of the development of geometric knowledge. The history of geometry, and the history of mathematics in general, contains numerous examples of creating and using wrong or inadequate items of knowledge, from our point of view of course. This deficiency of geometry and mathematics is usually hidden from the student who may, and frequently does, hold a belief that a work done by a mathematician is free from error, which is obviously not the case. By applying activity 1, geometry will certainly move from a position of absolute truth to that which has an evolving knowledge base for a given student.

What follows is a sample of wrong and inadequate items of knowledge which are mostly taken from Eves (1990). Each item is coupled with some questions and requirements which may be used in utilizing the activity.

- The area of a quadrilateral in Babylonian mathematics was determined by using the formula $K = (a + c)(b + d)/4$, where a, b, c, d are consecutive sides of the figure. Can this formula be used for some quadrilaterals? What about others? How might this formula have been obtained? How did ancient Egyptians calculate the area of plane figures? Which plane figures did they consider?
- The volume of a frustum of a squared pyramid in Babylonian mathematics was found by multiplying the solid's altitude and half the sum of its bases. Is this formula valid for some solids? How might this formula have been obtained? How can a correct formula be obtained? Find another wrong or inadequate formula for determining the volume of a solid.
- Ancient Chinese found the area of a circular segment of chord c and depth s by using the formula $s(c + s)/2$. How might this formula have been obtained? How can a correct formula be obtained? What was the nature of ancient mathematics?
- The area of a circle in ancient Egypt was calculated by taking the square of eight ninths of the circle's diameter (Robins & Shute 1987). What value is taken for π? How might this procedure have been obtained? How did ancient Egyptians calculate the diameter of a circle of a known area?
- Ancient Hindus solved the circle-squaring problem by using two formulas: $d = (2 + \sqrt{2})s/3$ and $s = 13d/15$, where s is the side of the square and d is the diameter of the equal circle. What value is taken for π in each of the cases? Why can't this problem be solved with straightedge and compasses? Can this problem be solved by using an additional tool? If so, present a solution. What are the three classical ancient problems? Examine them in fuller detail.
- Heron constructed a regular heptagon (seven-sided polygon) by taking for its side the apothem of a regular hexagon having the same circumcircle. Is this a good approximation? Can a regular heptagon be precisely constructed with straightedge and compasses? Justify your answer.
- The area of a circle in Byzantine mathematics was calculated by taking the geometric mean of the areas of the squares inscribed in and circumscribed about this circle (Heath 1981). What value is taken for π? What might have been the origin of this method? What value is taken for π if the arithmetic mean was used?
- Euclid used a fact that if a line passes through two points that are at different sides

of line *l*, it then has a point in common with *l*. (Kline 1980). Is something wrong with his argument? Did this fact follow from Euclid's axioms? Which two concepts were essentially missing? Why did Euclid commit this mistake? What was the impact of his mistake?

- Euclid's fifth postulate states that: "If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles." (Eves 1990, p. 28) Claudius Ptolemy (ca. AD 150) proved this postulate by assuming that through a given point not on a given line only one parallel line can be drawn to that line. Omar Khayyam (c. 1050–1123) verified this by recognizing that two lines perpendicular to a third line must be equidistant lines. John Wallis (1616–1703) justified Euclid's fifth by using the assumption that similar noncongruent triangles exist. Johann Lambert (1728–1777) based his proof of this postulate upon the fact that in a quadrilateral with three right angles the fourth angle must also be right. Adrien-Marie Legandre (1752–1833) proved the postulate by assuming that through a point within a given angle less than 60° a line intersecting both sides of the angle can always be drawn. Why were these mathematicians, and many others who did the same, wrong? Why did they try to prove this postulate? What was the impact of their work?
- In the first edition of his book "Foundations of Geometry" published in 1899, David Hilbert (1862–1943) proposed a postulate set containing a preposition which was later proved by E. H. Moore on the basis of Hilbert's other postulates. What does this note tell us? How does the knowledge of geometry seem to develop?

Examining "errors" should provoke a number of general questions such as: "What was known in the mathematics of that time?", "What do these mistakes tell us about the mathematics of that time?", "Why might these errors have arisen?", "How might these errors have been obtained?", "What might the impact of the errors have been?", and so on. Our experience suggests that answering such questions are beneficial for many students. Although the questions may not always be suitable for regular classroom work, they are quite suitable for project work, requiring students to collect and comment on errors for a given topic or a specified historic period, for instance.

Some errors may be examined in the following, less demanding way.

Teacher (T). Draw a circle. Draw its circumscibed and inscribed squares. Can you find the area of this circle by using the areas of these squares.

Student (S). Can we use half the sum of the areas?

T. Why not. Please try it out!

· · · some calculations are performed.

S. The area is 3 times *r* squared.

T. Is it good?

S. It' s acceptable as π is about 3.14.
T. Well. We used the arithmetic mean. Can we use other means? The geometric, or the harmonic?
· · · some calculations are performed.
S. The geometric mean is not acceptable.
T. Why not?
S. It assumes that π is, $\sqrt{8}$ which is obviously wrong.
T. That' s right, but please remember that this wrong approximation was used in Byzantine mathematics.

The following dialogue occurred in a ninth-grade class after an introduction regarding heuristic and deductive aspects of mathematics, which were metaphorically connected with building the house of mathematics and then visiting this built house. During this introduction we underlined, among other things, that even in mathematics an item of knowledge cannot be 100% certain.

S. (apparently astonished) Does this mean that we learn some things in school that may not be true?
T. In secondary mathematics probably not, but look at geography. You' ve just learned about the hypothesis of the origin of the Earth. Do you know what a hypotheses is?
S. (little confused) Yes. It' s some kind of a sound but not yet proved fact, isn' t it?
T. That' s right.
S. (surprised) Yes, I see. I get the point.
T. Please remember an old Latin saying "*Opinio magistri probabilis tantum*."
S. What does it mean?
T. It means that authority' s opinion, including mine, should only be taken as probably certain.

ACTIVITY 2 —Demonstrating the ways of creating and testing items of knowledge

Even in the mathematical sciences, our principal instruments to discover the truth are induction and analogy. —Pierre-Simon Laplace

Activity 2 deals with explaining how some pieces of geometry knowledge can be created and tested against certainty. The history of geometry evidences that geometric knowledge has been mostly created by using inductive and analogical reasoning, which are special cases of plausible heuristic reasoning. Despite that, most mathematicians, both researchers and educators, usually say nothing about how the considered items of knowledge have been derived and why. Furthermore, almost nothing is said about how these items of knowledge can be tested against certainty. This negative attitude creates a false view that items of knowledge have been discovered by "someone there" in the form in which they are presented to the audience. By applying activity 2, this view will surely change, making geometric knowledge more personal to the student.

"Finished mathematics presented in a finished form appears as purely demonstrative, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making. You have to guess a mathematical theorem before you prove it; you have to guess the idea of the proof before you carry through the detail." (Pólya, 1954 p. vi) What follows is a sample of examples regarding geometry in the making. Each example is coupled with some questions and requirements which may be used in utilizing the activity. Each group of examples is followed by some general comments relating to the teaching/learning process and/or relevant research findings.

- – Consider the sum of the interior angles of convex polygons. This sum is 180° for triangles. The sum is 360° for quadrilaterals. What about pentagons? Can you see a pattern? What is the sum for n-sided convex polygons. What about n-sided concave polygons?
  – Investigate the number of diagonals of convex polygons and disclose a pattern.
  – Consider various plane figures of equal perimeter. Which one seems to have the maximum area?
  – Examine the number of 1×1 squares intersected by a diagonal in a rectangular array. What seems to be the relation between that number and the size of a rectangle? (Pagni 1991).
  – Consider the number of faces, the number of vertices, and the number of edges of a convex polyhedron. Are these numbers related in some way? (Pólya 1954)
  Plausible reasoning is based upon inductive reasoning whereby we come up with a general pattern by examining some examples. Our early knowledge that addition is commutative and associative was acquired in that way, for example. It has been widely recognized that deductive reasoning cannot be cultivated without inductive reasoning. The NCTM 9–12 Standards, for example, explicitly emphasise the following.

  > "Inductive and deductive reasoning are required individually and in correct in all areas of mathematics. A mathematician or a student who is doing mathematics often makes a conjecture by generalizing from a pattern of observations made in particular cases (inductive reasoning) and then tests the conjecture by constructing either a logical verification or a counterexample (deductive reasoning). It is a goal of this standards [mathematics as reasoning] that all students experience these activities so that they come to appreciate the role of both forms of reasoning in mathematics and in situations outside mathematics." (NCTM 1989, p. 143)

- – The diagonals of a rectangle meet in a point bisecting each other. This is a theorem of plane geometry. Find a solid that is analogous to a rectangle, and state an analogous fact regarding solid geometry. Does the fact hold true?
  – Do the same for the following 2-D theorem: three angle bisectors of a triangle have a common point that is the center of the circle inscribed in that triangle.

– Find a solid that is analogous to an isosceles. Find a theorem regarding an isosceles and an analogous assertion regarding that solid. Is this assertion a 3-D theorem?
– Devise a geometrical method that is analogous to a numerical method of ancient times termed *regula falsi*. The equation: $x + x/3 = 12$ is solved with it by assuming that $x = 3$, which yields $x + x/3 = 4$. Since 12 is three times bigger than 4, x equals $3 * 3 = 9$ (Kadijevich 1990).

Heuristic reasoning is also based upon analogy. It deals with some sort of structural similarity between two knowledge domains. This means that the more structural similarities between two domains we can conceive, the more analogies between them we can establish, and therefore the more items of knowledge we can advance. Of course, using analogies, like using any form of heuristic reasoning, yields right as well as wrong items of knowledge. Research shows that students tend to use superficial analogies that are based upon surface or physical similarities of the domains, not upon their underlying structures or applied methods of solution (Kadijevich 1993).

• – The area of a triangle is half the product of its base and altitude. A circle can be considered as a union of infinite number of equal congruent triangles, each of which has infinitely short base and altitude equals to the radius of that circle. The area of the circle is then the sum of the areas of the triangles. This area is thus half the product of the circumference of the circle and its radius. Does this kind of reasoning yield the correct formula?
– What formulas are suggested by applying this method to determining the area of the lateral surface of a right circular cone and the volume of a sphere?
– Does the method seem pragmatically valid? Is the method logically correct? Why or why not?

The method is established by the analogy between the infinite and the finite. A very able learner may realize that the applied method, which is based upon the use of infinitesimal arguments, is logically wrong. Our experience evidences that this rarely occurs, even when students are explicitly required to question the validity of the method, most probably because the majority of students believe that a method yielding correct results is valid as well. The applied method is still wrong. Indeed. The base of the triangle in question is either zero or greater than it, which implies that its area is either zero or greater than it. If it is zero, the area of the circle is also zero, which is wrong. If it is greater than zero, the area of the circle is infinite (we add infinitely many equal terms), which is wrong too. As an exercise, find out the infinite sum: $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ and then come up with a valid version of the method by using non-congruent triangles. Note that the described method is logically grounded by non-standard analysis, however (Davis & Hersh 1990a).

- – The area of a rectangle is the product of its sides. Consider a rectangle and a parallelogram with equal bases and altitudes. Let the figures lie between two parallel lines. Draw any line parallel to these lines that crosses the figures. Since such a line cuts off line segments of equal length, the areas of these figures should be the same. We may imagine that infinite number of equal line segments are first piled vertically to produce the first figure, and then piled obliquely to produce the second one. Does this kind of reasoning yield a correct formula?
  - Apply this method to other examples in 2-D. Formulate the underlying principle.
  - Apply this method to 3-D through finding the volume of a oblique prism by using a right prism. What about pyramids? What is the underlying principle now?
  - What does the principle suggest for the volume of a sphere? (Use four solids: a sphere, a cylinder circumscibed around it and two cones whose bases coincide with those of the cylinder and whose vertices coincide with the center of the sphere (Sharygin 1997)).
  - Does the method seem pragmatically valid? Is the method logically correct? Why or why not? Can you prove it by using integration?

  This infinitesimally-based method, established again by the analogy between the infinite and the finite, is the well-known Cavalieri's principle, first formulated in 1629. Probably it was widely used first by Archimedes (III century BC) who disclosed it in his book *The Method*, realizing that "the assumption that areas (or volumes) can be envisaged as the aggregate of a large number of line segments (or areas), while useful for discovering theorems, is not acceptable in a rigorous proof." (Hollingdale 1989, p. 79).

- – Plausible reasoning has its own patters of inference. One of them, the so-called *heuristic modus ponens*, says: if *A* implies *B*, and *B* is true, then *A* is more credible. In other words, a conjecture (*A*) becomes more credible if its consequence (*B*) is verified. By using this pattern, examine an item of knowledge that states the following: the area of a trapezoid is obtained by multiplying its altitude and half the sum of its bases. Consider a triangle and a parallelogram that may be regarded as special cases of a trapezoid.
  - By using the heuristic modus ponens, what can we say about the validity of the following assertion: the area of the lateral surface of the frustum of a right circular cone is $p\,(R+r)\sqrt{(R-r)^2+h^2}$, where R is the radius of the solid's lower base, r is the radius of its upper base and h is the solid's altitude? What special cases of the frustum may be considered? Which pattern of reasoning is in fact used when several different consequences are justified? Is this pattern related to inductive reasoning?
  - Examine other patterns of plausible inference suggested by Pólya (1954) and give some examples of their application. Which pattern seems to underlie our everyday

reasoning?

– Note that geometrical and physical formulas can quickly be checked by using the test by dimension (Pólya 1990).

Patterns of plausible inference prove nothing. They are however useful in creating new pieces of knowledge. This is because their application points out to items of knowledge deserving further study. Let *A* implies *B*. By using the heuristic modus ponens, our confidence in a conjecture (*A*) increases when we justify its conesquence (*B*), especially if the consequence is very improbable in itself. The so-called *heuristic modus tollens* says: if *A* implies *B* and *A* is false, then *B* is less credible. By using it, our confidence in a conjecture (*B*) decreases when we disprove a possible ground for that conjecture (*A*), especially if the ground is very probably in itself.

The following dialogue suggests how activity 2 may be utilized.

T. How do we calculate the area of a trapezium?
S. We multiply its height with half the sum of its bases.
T. Fine. What may the formula for the volume of a truncated squared pyramid be?
S. Multiply its height with half the sum of its bases.
T. Why?
S. Because of analogy. This solid becomes a trapezium in 2 D,.
T. That's right. Can you check the formula?
S. Its dimension is Ok.
T. Could you explain a bit.
S. The dimension of each of the areas is two, thus the dimension of half the sum of them is again two. The dimension of height as any length is one. Since we multiple these quantities, the dimension of the product is 3, which is the dimension of the volume as well.
T. Fine. Can you check the formula further?
S. Hm.
T. What if the solid becomes a prism?
S. Well, the bases are of the same area. The formula then says "multiply the solid's height with its base". It is true for the volume of a prism. It seems that the formula is right.
T. Test further.
S. When the solid becomes pyramid, the formula says "multiply the solid's height with half its base", which is obviously wrong.
T. Well done. What can we finally say?
S. The formula is wrong.
T. This wrong formula was however used in the Babylonian mathematics. Here is the right formula. Come up with a computer program comparing the outcome of these formulas for different examples.

Items of geometry knowledge can be created and tested by using computer software such as the "Geometric Supposer" (Schwartz & Yerushalmy 1985). This program encourages students to make and test their own conjectures as follows. The student undertakes some measurements and constructions on a simple example of an object. The program records all of these actions. He/she then makes a conjecture. This conjecture is then tested by requiring the program to apply the recorded actions to various examples of the same object. The student specifies these examples by using formal geometric language. The evidence showed that the work with the "Geometric Supposer" is beneficial for students although it demands hard and somewhat frustrating work from both students and their teachers (Clements & Battista 1992). The "Geometric Supposer" has therefore been developed further into the "Geometric supperSupposer" (Yerushalmy & Schwartz 1993). Other programs encouraging the making and testing of conjectures are "Cabri-géomètre" (Bellemain & Laborde 1994) and the "Geometer's Sketchpad" (Jackiw 1992). Their drag-mode allows the student to move certain parts of a figure without changing its underlying geometrical relationships. A four-year experience in utilizing Cabri I (Hölzl 1996) shows that the drag-mode changes the traditional status of points and lines and promotes new styles of reasoning. This experience also shows that the shift of importance from figure construction to figure investigation does need meaningful experiments, the creation of which may indeed be a hard, frustrating enterprise for the learner. Despite that, the work with the just reported or other similar programs does change the traditional view of geometric knowledge, making it more personal for skillful experimentalists.

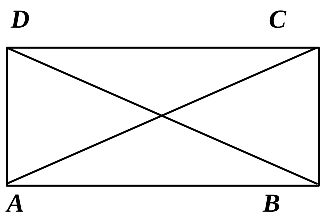
ACTIVITY 3 —Considering proving as a form of social interaction

Mathematics in real life is a form of social interaction where "proof" is a complex of the formal and the informal, of calculations and casual comments, of convincing argument and appeals to the imagination. —Philip J. Davis & Reuben Hersh

Activity 3 is about considering proof as a form of social interaction. Mathematical knowledge, as knowledge in general, has both individual and societal features. This is because every item of knowledge has evolved from inter-subjective agreements on conceptions of individuals (Ernest 1991). Despite that, mathematical teaching rarely promotes this socio-constructivist nature of mathematics. As a result, most students, even at a high level of mathematical education, cannot perceive doing mathematics as a form of social interaction occurring in the real world (Davis & Hersh 1990b). It is therefore not surprising that many students hold a wrong view about the nature of proof, which according to Clements & Battista (1992) negatively influences their proof skills. By applying activity 3, things will certainly change, making geometrical proofs more accessible to the student.

The socio-constructivist nature of proof assumes, among other things, the following: (a) proof is a result of both individual construction and social negotiation, (b) proof is an argument convincing those who know the subject, and (b) proof involves both formal and informal arguments. Activity 3 is therefore based upon negotiation including both formal and informal geometric language. This negotiation can occur between proof builders/presenters and proof examiners. The proof examiners, who should not be involved in proving/presentation, ask various questions such as: "Can you simplify your argument?", "Can you exchange some arguments?", "Does you argument hold true in general?, etc. The proof builders/presenters clearly expose their arguments and answers, keeping in mind the following fact: the student' s "task is not only to find a solution that is personally satisfying but to write up a solution that would satisfy a reader. Mathematics is about communication" and thus the student "who cannot communicate what he or she has done with a problem has not truly solved it" (Kilpatrick 1992, p. 43). The presented negotiation aims at the realization of the relative position of proof and its content. To achieve this end, both the builder/presenter and the examiner need to be aware that proving is successfully done when shared meanings between them have been established. Of course, some students must be previously convinced that the purpose of the proof is established by the public status of knowledge and that proving is not something that has already been done for them by others. (Some teacher' s guidance regarding these issues may indeed be required.)

Arguments can be presented within a two-column proof, the left column of which contains certain claims, while the right column briefly gives the basis for them. This kind of presentation like the one given below (Fig. 1) does clarify the process of deductive reasoning since it requires the student to (a) specify what is given and what is wanted-some students frequently use wanted instead of given; and (b) support his/her claims with clear arguments, gradually approaching what is wanted.



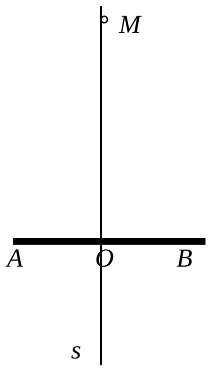
Given: $AB = DC$, $AD = BC$ $\angle A = \angle B = \angle C = \angle D = 90°$	Wanted: $AC = BD$
CLAIM	ARGUMENT
$AB = BA$	= is symmetrical
AD = BC	Given
$\angle A = \angle B$	Given
$\Delta ABD \cong \Delta BAC$	*SAS* rule of triangle congruency
$AC = BD$	a consequence of this congruency wanted

Figure 1. The process of deductive reasoning

Activity 3 can be utilized by using three types of tasks. The first type, such as task 1 below, requires students to reassemble a complete proof whose parts are given, and to make its variations if some lines of reasoning may be interchanged. (The given parts may contain some wrong or useless elements that need to be excluded from the proof reconstruction.)

The second type, like task 2 below, asks students to evaluate a given proof, and to suggest its correction/improvement if need be. The third type, such as tasks 3 and 4 below, requires students to create several proofs for the same assertion.

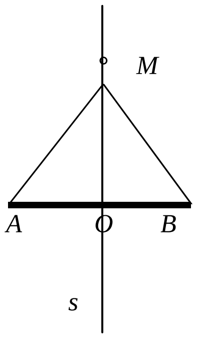
1. By using the following cards (Fig. 2), reassemble a proof. What assertion is justified by this proof? Are there some variations of the proof?



Given: $AO = OB$, $\angle AOM = \angle BOM$	Wanted: $AM = BM$
CLAIM	ARGUMENT
$OM = OM$	Given
$AM = BM$	*SAS*
$\angle AOM = \angle BOM$	Identity
$\Delta AOM \cong \Delta BOM$	*SAS* consequence
$AO = OB$	Given
$AB = BA$	Identity

Figure 2. The proof cards

2. Fig. 3 presents a proof of this assertion: if a point is equidistant from the endpoints of a line segment, it then belongs to the perpendicular bisector of that segment. Is this proof correct? Why? Correct/improve it. Give some variations for a revised proof.



Given: $MA = MB$	Wanted: AB is symetric with respect to line s
CLAIM	ARGUMENT
$MA = MB$	given
$AO = BO$	O is the midpoint of AB
$OM = OM$	identity
$\Delta AOM \cong \Delta BOM$	*SSS*
$\angle AMO = \angle BMO$ (1)	*SSS* consequence
ΔABM is symetric with respect to line s (2)	the consequence of (1)
AB is symetric with respect to line s	the consequence of (2); wanted

Figure 3. A proof of assertion

3. Demonstrate that the equality of the base angles of an isosceles triangle can be proved by applying alone any of the following congruency rules: SSS, SAS, SSA. How many different justifications can be based upon the SSS rule? Why can only one justification be grounded on either SAS or SSA? Why is the use of ASA ruled out?
4. Prove that the diagonals of a rhombus are bisectors of its angles by using the following fact: (a) a rhombus is a parallelogram with all sides equal in length; (b) the diagonals of a rhombus bisect each other; and (c) the diagonals of a rhombus bisect each other at right angle. Find as many different proofs as you can. Variations of the same proof may also be examined.

   Of course, tasks need not be restricted to the area of triangle's congruency. The teacher may also include some non-standard tasks, such as task 5 below derived from Davis & Hersh (1990b).
5. By using the following facts and rules, prove that every sportsman is a member of at least two sport clubs. FACTS: A sport club is a union of one or more sportsmen. A sportsman in a sport club is a member of this club. Two sport clubs are joined if every member of one of them is also a member of the other, and vice versa. Two sport clubs are separated if they have no members in common. RULES: Every sportsman is a member of at least one sport club. Every two sportsmen are members of exactly one sport club. Every sport club has exactly one separated sport club.

Although many students may need some teacher's guidance in utilizing activity 4 successfully, tasks on reassembling a given proof are not likely to require particular guidance. According to Goldstein (1989), these tasks seem suitable and beneficial for most students, and hence should be primarily used. As tasks on evaluating a given proof may be demanding for most students, incremental guidance should be planned beforehand, and applied if need be. For example, students may be told that two lines of reasoning are wrong or that the above-mentioned confusion between given and wanted is present. As regards a task on justifying an item of knowledge in several ways, considerable guidance may indeed be needed. Furthermore, there is another problem: solving such tasks is unlikely to be the main concern of most students, even the able ones. Things may change, however, if extra credit is given for additional solutions, e.g. three different solutions of one task may be scored as three single solutions of three different tasks. It is important to underline that solving tasks in more then one way is a very important learning activity since it promotes both skills and understanding. According to the NCES document "Pursuing Excellence" relating to the Third International Mathematics and Science Study (NCES 1996), applying this activity was a distinctive feature of Japanese mathematical teaching, which enabled Japanese students to obtain considerably higher test scores than U.S. and German students who were taught in the traditional way emphasising skills rather than understanding.

Activity 3 can also be utilized by using tasks requiring students to formulate and then prove or disprove conjectures. Solving such tasks may be based upon the negotiation between item developers and item examiners. These tasks may be related to little axiomatic systems like the one regarding the properties of parallelograms (NCTM 1989, pp. 159–160). Non-standard systems may also be used such as the one about finite geometries requiring students to apply both algebraic and analytic geometry knowledge (Cofman 1995). Such unfamiliar, provoking tasks may particularly encourage student to make, test and justify their conjectures. Research shows that this "make, test and justify conjecture" approach seems beneficial for students (Clements & Battista 1992). The approach may be based upon Lakatos's model for the heuristics of mathematical discovery (Lakatosh 1976). Although this model does not seem relevant to mathematical knowledge in general (Hanna 1997), it is becoming clear that its empirical values are heavily influenced by a teacher's guidance skills (e.g., Balacheff (1991)). It therefore seems that, no matter which model is used, the more the item developer and the item examiner are epistemologically skillful, the more the approach is successfully applied.

ACTIVITY 4—Examining the use of items of knowledge in modelling the reality

The purpose of mathematics education should be to enable students to realize, understand, judge, utilize and sometimes also perform the application of mathe-

matics in society. —Mogens Niss

Activity 4 is based upon considering various examples relating to the use of geometry in modelling reality. Although the knowledge of geometry aids the solutions of many problems of everyday life such as building houses, packaging, advertizing, planning a sport field, constructing tunnels and bridges, and working out city and road maps (Graumann 1989), it seems that many students are not aware of this great fact. This is because geometric teaching does not promote enough activities regarding application and modelling like examining eclipses via circles, reflections via conic sections, sound waves via hyperbolas, and tiling via regular polygons (Fremont 1979); or designing a rear windscreen wiper (Clatworthy & Galbraith 1991); or analyzing a water supply from a local lake (Matsumiya, Yanagimoto & Mori 1989). It is therefore important to infuse curricula with "examples of how geometry is used in recreations (as in billiards or sailing); in practical tasks (as in purchasing paint for a room); in the sciences (as in the description and analysis of mineral crystals); and in the arts (as in perspective drawing)" (NCTM 1989, p. 157), for instance. By applying activity 4, the knowledge of geometry will certainly become alive for the student, who will begin to perceive geometry as a human enterprise which improves our lives.

Activity 4 can be utilized by using three types of tasks. The first type requires students to find out why the presented ideas work by pointing out the relevant items of underlying geometric knowledge, such as tasks 1–3 given below. Of course, in case of complex ideas, the request for disclosing and justifying the underlying knowledge may be omitted. The second type asks students to come up with suitable applications of some items of geometric knowledge, thus bringing these items to life. The area of application may or may not be given. Tasks 4–5 given below are of this kind. The third type requires students to perform the a complex application of geometric knowledge through modelling activities, like tasks 6–7 below. This kind of task may primarily be used in project work, since modelling activities call for a complex interplay among cognitive, metacognitive and affective domains (Kadijevich 1993) which seems to be beyond the competence of most students. (As regard modelling in general, several valuable references like Swetz & Hartzler (1991) are available now.)

1. A rectangular foundation is usually marked by a rope, and its digging is undertaken if the diagonals of the rectangle are (almost) equal. Otherwise, the rope is stretched again and the check is repeated. Which item of knowledge validates this procedure?
2. A Pythagorean triangle with sides in the ratio 3 : 4 : 5 can be used to check if a square is properly doubled or tripled. The acute angle between the diagonals of a doubled (tripled) square must be equal to the bigger (smaller) acute angle of the triangle (Hollingdale 1989). Check this method theoretically.

3. The current position of a ship on the sea can be determinated, among other methods, by measuring from it two horizontal angles between three objects at the coast that are represented on a map. This is because these objects and the obtained angles allow us to construct on the map two circles, one intersection of which defines the desired position (Gardner 1987). Which item of knowledge enables this kind of navigation?
4. A transversal intersects two parallel lines so that the alternate-interior angles are equal. Apply this item of knowledge to make an optical instrument. (For a solution, see Fremont (1979)).
5. A hyperbola is the locus of points whose distances from two fixed points (the loci) have a constant difference. Apply this item of knowledge to a real-world situation. (A solution is given in Fremont (1979)).
6. Design a rear windscreen wiper. (A solution is provided by Clatworthy & Galbraith (1991)).
7. Analyze a water supply from a local lake. (A detailed analysis is presented in Matsumiya, Yanagimoto & Mori (1989)).

During last summer we spent a week in a mathematical summer school teaching some non-standard topics. Among them was "the elements of seashore navigation" based upon elementary geometric knowledge. The reactions of two students are noteworthy. While one angrily commented that this topic has not mathematics at all, the other delightedly said that it was the first time he realized the practical value of the considered geometric knowledge. As this experience shows, students may view our teaching in quite different ways that are certainly influenced by their conceptions of mathematics.

CLOSING REMARKS

It is widely known that the more adequate a student's conceptions of mathematics and its teaching/learning are, the more knowledge he/she is likely to acquire and apply successfully. This article proposed four activities whereby an adequate, humanistic picture about geometric knowledge can be promoted. Although a rich treasure of findings concerning the teaching and learning of geometry has emerged in the last two decades (e.g., Hershkowitz (1990) and Clements & Battista (1992)), it seems that designing and studying the proposed activities have so far been neglected by most researchers in the field. It is true that, for example, the NCTM 9–12 Standards (NCTM, 1989) provide a number of requirements relating to activities 1–4. However, this document, like most others, fully emphasises only applicative and structural issues of geometric knowledge. It still does not particularly deal with the historical and epistemologycal issues raised by this article. There is no doubt that utilizing the proposed activities will improve the practice of teaching as well as promote better learning abilities. It is therefore important to develop these activities further and study their empirical values in more detail.

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