Djordje Kadijevich
Improving mathematics education: neglected topics and further research directions

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# Abstract

Various evaluations around the world evidence that school mathematics is in need of improvement. An inquiry into post-14 mathematics education in England, Wales and Northern Ireland, for example, underlines, among other things, its failure to meet not only the needs of many learners but also the expectations of employers and higher education institutions.

Such an improper state of mathematics education has primarily been caused by inappropriate teaching. Although mathematical proficiency has many interwoven and inter-dependent aspects — e.g. understanding, computing, applying, reasoning and engaging — mathematics teachers have tended to focus on one aspect at a time, wrongly hoping that other aspect(s) would (spontaneously) develop as a consequence.

Ten years of my research in mathematics education have revealed four neglected topics, the theoretical and empirical elaborations of which would improve the field and promote better teaching. These topics are: (1) promoting the human face of mathematics; (2) relating procedural and conceptual mathematical knowledge; (3) utilizing mathematical modelling in a humanistic, technologically-supported way; and (4) promoting technology-based learning through applications and modelling, multimedia design, and on-line collaboration.

Mathematics does have a human face and, to promote this, we may utilize the following activities: (1) examining wrong and inadequate items from the phylogenesis of mathematical knowledge; (2) demonstrating ways of creating and testing items of this knowledge; (3) considering proving as a form of social interaction, and (4) examining the use of items of mathematical knowledge in modelling reality. Although many articles deal with humanistic mathematics education, the need for design and use of such or similar activities has not been fully realized so far by most researchers in the field. The first part of this dissertation exemplifies these four activities, suggests how to develop them further, and indicates which of their empirical values may be studied.

Although developing and relating procedural (*P*) and conceptual (*C*) mathematical knowledge is a very important goal of mathematics education, it is rarely attained, especially when their relation is in focus. However, only a few studies have examined *P*-*C* links in detail. In part two, among other things, I examine several views on the relation between the two knowledge types, describe four theoretical constructs whereby establishing *P*-*C* links can be explained, and report a successful operationalization of two of these constructs. Part two also considers important research questions for further studies focusing on sequencing/combining procedures-first and concepts-first teaching/learning, critical variable(s) influencing *P-C* link, and the impact of different technologies (i.e. learning opportunities of different technologies) on this link.

Despite their unquestionable educational value, applications and mathematical modelling have so far played a marginal role in everyday mathematics education mostly. To overcome such an inappropriate state, we should help mathematics educators (teachers, teacher trainers and policy makers) realize the full power of computer-based modelling, develop suitable standards of such modelling, and ensure their proper utilization. Having applied a humanistic context, the third part of this dissertation justifies these three goals, explains how they can be attained, indicates a direction into which the presented standards may be elaborated, and suggests terms in which the utilization of these standards in day-to-day practice should be examined.

To have more opportunities for learning mathematics, students and pre-service and in-service teachers should be required to do applications and modelling with versatile technological tools, to design multimedia lessons, and/or to create on-line collaborative works. The last part of the dissertation exemplifies these learning activities and gives directions for further research including the main stages of instrumental genesis for a particular tool, interactive multimedia features that maximize learning efficiency, and the theoretical frameworks of Internet-based learning regarding the design of mathematical problems and comprehension modelling tools.

# List of original publications

This dissertation is based on the following seven papers, which are referred to in the text by their Roman numerals (**I**-**VII**).

**I.** Haapasalo, L. & Kadijevich, Dj. (2000). Two types of mathematical knowledge and their relation. *Journal für Mathematik-Didaktik*, **21**, 2, 139-157.

**II.** Kadijevich, Dj. (1998). Promoting the human face of geometry in mathematical teaching at the upper secondary level. *Research in Mathematical Education*, **2**, 1, 21-39.

**III.** Kadijevich, Dj. (2002). An Internet-based collaborative environment for the learning of mathematics. *Journal of Computer Assisted Learning*, **18**, 1, 48-50.

**IV.** Kadijevich, Dj. (2004). How to attain a wider implementation of mathematical modelling in everyday mathematics education? In H.-W. Henn & W. Blum (Eds.), *ICMI Study 14: Applications and Modelling in Mathematics Education* (pp. 133-138). Dortmund, Germany: University of Dortmund.

**V.** Kadijevich, Dj. & Haapasalo, L. (2001). Linking procedural and conceptual mathematical knowledge through CAL. *Journal of Computer Assisted Learning*, **17**, 2, 156-165.

**VI.** Kadijevich, Dj. & Haapasalo, L. (2004). Mathematics teachers as multimedia lessons designers. In J. B. Lagrange, M. Artigue, D. Guin, C. Laborde, D. Lenne & L. Trouche (Eds.), *On-line Proceedings of the ITEM Conference, Reims, June 2003*. Institut Universitaire de Formation des Maîtres de l'Académie de Reims: www.reims.iufm.fr/Recherche/ereca/colloques/.

**VII.** Kadijevich, Dj., Haapasalo, L. & Hvorecky, J. (2004). Using technology in applications and modelling (paper presented in the TSG 20 "Mathematical applications and modelling in the teaching and learning of mathematics" at the 10th International Conference on Mathematical Education, Copenhagen, Denmark, July 4-11, 2004). Available at www.icme-10.dk (to appear in *Teaching Mathematics and its Applications*).

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Finally my sincere thanks go to my parents Miloš and Natalija and sister Olja who have strongly supported my research all these years. I dedicate this dissertation to them.

In Belgrade, on the Day of St. Luke the Evangelist, 2004

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# 1 Introduction

My research in the early 1990's concerning lower levels of mathematics education revealed that teachers frequently fail to teach mathematics in an appropriate way (Kadijevich, 1993). Have matters been improved so far?

According to Kilpatrick and Swafford (2002), many students are not particularly successful in developing and applying complex computational skills and they do not demonstrate much understanding of mathematical concepts utilized in calculations and problem solving. Although this finding portrays teaching/learning outcomes in the USA (probably mostly at primary and lower secondary levels), there is little doubt that such a finding is relevant to many (perhaps most) mathematical students around the world, especially at lower levels of mathematics education.[1]

Let us give two examples supporting the claim that, because of instructional practice, learning approach, test item types and/or other reasons, both skilled algorithmic performance and genuine understanding are typically not demonstrated simultaneously by students for examined topics. In Finland, one of highest-achieving countries in the PISA 2000 study, students performed rather poorly at tasks requiring generalizations and explanations[2]

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[1] Younger students tend to perform better on traditional tests than older ones (see, for example, www.erzwiss.uni-hamburg.de/IGLU/home.htm). It seems that the older the students are, the more negative the impact of instruction on them is, which is in accord with Skemp (1987) who found that "Not only do we fail to teach children mathematics, but we teach many of them to dislike it." (p. 3)

[2] A high overall performance of the Finns may be explained as the outcome of the assessment tasks that strongly emphasized the use and applications of mathematical knowledge, which, along with problem solving, have played a

(see Kupari, 2003). In Japan, one of TIMSS highest-achieving countries, an analysis concerning the TIMSS 1995 and previous studies, revealed that attainment levels concerning problems requiring a higher degree of comprehension and thinking could not be regarded as acceptable (Sawada, 1999). An unfavourable state of mathematics education can also be found at the upper secondary level. For example, Smith's (2004) inquiry into post-14 mathematics education in England, Wales and Northern Ireland underlines, among other things, its failure to meet not only the needs of many learners but also the expectations of employers and higher education institutions.

The cited studies are not concerned with the same educational level. They do not apply the same method concerning the quality of teaching/learning outcomes. However, they do provide some evidence that students' knowledge of mathematics has not been acquired to a degree expected by the community of mathematics educators and/or employers in the above-mentioned countries.[3] I thus agree with Ralston (2004), who found that school mathematics is in need of improvement.[4]

Why is the general state of school mathematics often unfavourable?

Although mathematical proficiency has many interwoven and interdependent aspects — e.g. understanding, computing,

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central role in the Finnish mathematics teaching since the middle of the 1980's.

[3] Such shortcomings have caused revisions of mathematics curricula in many countries around the world. In Japan, for example, primary students now learn a new subject, *Integrated Learning*, to grasp methods of thinking and learning and to be able to pose and solve problems themselves (Tanaka and Wong, 2000).

[4] Various shortcomings of teaching and learning mathematics exist at university level. Main of them have been generated by an inappropriately treated qualitative changes, reconstructions and cognitive flexibility in mathematics learning (Artigue, 1999). An analysis of the state of university mathematics education and a range of possibilities for its improvements can, for example, be found in Holton (2001).

applying, reasoning and engaging[5] — mathematics teachers have tended to focus on one aspect at a time, wrongly hoping that other aspect(s) would (spontaneously) develop as a consequence (Kilpatrick and Swafford 2002). And, even when teachers use good problems (e.g. those that focus on concepts and connections among mathematical ideas), they can implement them in a wrong or inadequate way, e.g. as problems that call for basic computational skills and procedures (see Stigler and Hiebert, 2004). Having in mind these issues and the above-presented teaching/learning outcomes, it seems that, despite possible simplifications, this unfavourable state of school mathematics has primarily be caused by inappropriate teaching that ignores personal, professional and societal needs.

Ten years of my research in mathematics education have revealed four neglected topics, a wider appropriate implementation of which would improve the traditional teaching.[6] These topics are:

- promoting the human face of mathematics,
- relating procedural and conceptual mathematical knowledge,
- utilizing mathematical modelling in a humanistic, technologically-supported way, and
- promoting technology-based learning through applications and modelling, multimedia design, and on-line collaboration.

Each of these topics is discussed in a separate chapter, each comprising the following parts: topic realization, topic consideration and topic reconsideration. These parts respectively deal with:

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[5] seeing mathematics as a useful, sensible and doable enterprise and being willing to work on it

[6] It is important to underline that, as Artigue (1999) remarked, substantial improvements of the traditional teaching cannot be achieved by easy and inexpensive means without a strong institutional support and a substantial positive change in teachers' knowledge, engagement, and day-to-day practice.

- my motivation for examining this topic,
- the outcome of my published study/studies, and
- my further considerations of the topic including relations to the work of others not reported in the utilized study/studies and directions for further research.

When available, the citation of my work is also given.

# 2 Promoting the human face of mathematics

## 2.1 Topic realization

As a scholar of the Danish Ministry of Education and the Danish Research Academy I spent a year (1992-93) at the Department of Computer Science, University of Copenhagen, where I studied the possibilities of the computer-assisted learning of mathematics. During my stay I attended a series of lectures on mathematics and society at IMFUFA, Roskilde University, given by Philip Davis, a distinguished mathematician from Brown University, Providence, USA. In one of these lectures, *The Five Types of Mathematical Educators*, Professor Davis proposed that mathematics education should display primarily the human face of mathematics rather than platonic, or algorithmic, or formalistic, or computerized, or mechanical face of the subject. Convinced that mathematics has a human face (see Davis, 1993), I was concerned with the question "How may this face be promoted?". In 1996, in a lecture at a seminar on the history and didactics of geometry involving about one hundred upper secondary mathematics teachers in Serbia, I briefly presented four concrete learning/teaching activities whereby the human face of the subject may be promoted (Kadijević, 1997). Then, I examined these activities in more detail in an article published by Korea Society of Mathematical Education (see **II**).

## 2.2 Topic consideration

To promote the human face of mathematics, we may utilize the following activities:

- examining wrong and inadequate items from the phylogenesis of mathematical knowledge,
- demonstrating ways of creating and testing items of this knowledge,
- considering proving as a form of social interaction, and
- examining the use of items of mathematical knowledge in modelling the reality (see **II**).

These activities are examined in the following sections by using various items of geometric knowledge. Although some of these items may be relevant to the lower secondary level, the examined activities, especially the first three, are primarily intended for mathematical teaching at the upper secondary or tertiary level[7].

### 2.2.1 Examining wrong and inadequate items from the phylogenesis of mathematical knowledge

The history of geometry, like the history of mathematics, contains numerous examples of creating and using wrong or inadequate pieces of knowledge (e.g. Kline, 1980), from our point of view of course. This deficiency of geometry and mathematics is usually hidden from the student who may, and frequently does, hold a belief that a work done by a mathematician is free from error, which is obviously not the case. To help students form a more adequate picture of the subject, the teacher should present to the students wrong and inadequate items from the phylogenesis of

[7] especially for teacher training

mathematical knowledge and promote appropriate discussions about them. Three examples for this activity are given below.[8]

- The area of a circle in ancient Egypt was found by taking the square of eight ninths of the circle's diameter (Robins and Shute, 1987). How might this procedure have been obtained? Can we speak about calculation or approximation? What value is taken for $\pi$?
- Euclid used a fact that if a line passes through two points that are at different sides of line *l*, it then has a point in common with *l* (see Kline, 1980). Is something wrong with his argument? Did this fact follow from Euclid's axioms? Were there essentially some concepts missing?
- Can a curve be defined as a geometric figure generated by a moving point? It still can not. In 1890 Peano proved that a moving point can pass through all points of a square (Hahn, 1968). What does this discovery tell us?

### 2.2.2 Demonstrating the ways of creating and testing items of mathematical knowledge

The history of geometry evidences that geometric knowledge has been mostly created by using inductive and analogical reasoning, which are special cases of plausible (heuristic) reasoning having its own patterns of inference such as *inductive pattern* — if *A* implies *B*, and *B* is true, then *A* is more credible, and *analogical pattern* — if *A* is analogous to *B*, and *B* is true, then *A* is more credible (see Pólya, 1954).[9] Despite that, most mathematicians usually say nothing about how the considered items of

[8] Contrast the first example of an empirical mathematics with the other examples of a deductive mathematics.

[9] Although these prove nothing, they are useful in creating new pieces of knowledge as their application points out to items of knowledge deserving further study.

knowledge have been derived and why, and how these may be tested against certainty. This negative attitude creates a false view that items of knowledge have been discovered by "someone there"[10] in the form in which they are presented to the audience. By demonstrating the ways of creating and testing items of mathematical knowledge and by letting students create and test such items in a traditional and/or technology-based learning environment[11] themselves, this inappropriate view may change, making geometric knowledge more personal to them. Three examples relevant to this activity are presented below.

- *Creating by analogical reasoning*. The assertion "The diagonals of a parallelepiped meet in a point bisecting each other." is analogous to the assertion "The diagonals of a rectangle meet in a point bisecting each other.". As the latter assertion holds true, by utilizing analogical pattern, the former one may apply too[12], establishing a ground to search for its proof.
- *Creating by inductive reasoning*. On the basis of an experiment involving a square, a rectangle and a trapezoid made up of three equilateral triangles (or on an experiment in a dynamic geometry environment[13] involving various quadrilaterals inscribed in a circle), by utilizing inductive pattern, one can

[10] at a place (e.g. the world of Platonic ideas) unreachable for the majority of students

[11] Such an environment should support heuristics reasoning through exploration and visualization. Dynamic geometry software may be a good choice if it makes use of carefully designed tasks and activities providing "opportunities for students to notice details, to conjecture, to make mistakes, to reflect, to interpret relationships among objects, and to offer tentative mathematical explanations." (Hanna, 2000; p. 21).

[12] Analogy deals with some sort of structural similarity between two knowledge domains. Its application yields right as well as wrong items of knowledge. Students tend to use superficial analogies that are based upon surface or physical similarities of the domains, not upon their underlying structures or applied methods of solution (Kadijevich, 1993).

[13] e.g. Geometer's Sketchpad (see www.keypress.com/sketchpad/)

claim that the sum of opposite interior angles of a quadrilateral inscribed in a circle is always 180°, having a ground to search for a proof of this relation.

- *Testing by specialization.* Since one's claim that the area of a trapezoid is found by multiplying its altitude and half the sum of its parallel sides[14] holds true for some special cases of a trapezoid like a triangle and a parallelogram, by utilizing inductive pattern we can increase our belief in the validity of this formula. Of course, specialization may also yield a counterexample disproving the tested item.

### 2.2.3 Considering proving as a form of social interaction

According to *Principles and Standards for School Mathematics* (NCTM, 2000), constructing mathematically appropriate arguments, judging such arguments, and communicating these to peers, teachers and others are the central issues of the high school curriculum, and a priority should be given to the production of logical arguments and their effective presentation rather than to the form of developed/examined proof (two-column, flow or paragraph proof). However, "the crucial question should not be whether students' arguments are expressed in a form that a logician would approve but whether they are adapted to the nature of the mathematical objects that a community of knowers wants to know more about." (Herbst, 2002; p. 308).

Because of the tradition or other reasons, the teacher/student may prefer one proof presentation form to the others.[15] But, no

[14] obtained, for example, from the fact that a diagonal of this figure divides it into two triangles having the same attitude

[15] For learning proving, flow proof (a tree structure proof illustration) clarifying the whole reasoning space may be more suitable than the other forms (see Anderson, 1995).

matter which proof presentation form is utilized, the student should realize that the order of some steps may be irrelevant (the "inputs" to the SAS rule in Figure 1 could be differently ordered), which means that several variations of the same proof may be possible. He/she should also realize that the claims and arguments of a concrete proof — being dependent on the common knowledge of the prover and the listener (reader) that may differ from case to case — may not be fixed. Proofs are thus not monolithic structures as textbooks usually portray. As mathematical proofs are forms of social interaction having both formal and informal features (Davis and Hersh, 1990), geometry teaching should, whenever possible, treat its proofs in this way.[16]

It has been widely accepted that, as the previous activity suggests, explorations precede proofs whenever reasonable. Empirical verifications may also follow the proof if the student looks at the world through the lenses of an experimental scientist who usually does not accept a fact derived from a theoretical justification without an empirical confirmation (Hanna, 2000).

**Given:** CD is the perpendicular bisector of AB
**Prove:** Δ ABC is isosceles

AD = BD (bisector) → ∠ ADC = ∠ BDC (perpendicular) → CD = CD (reflexive) →

Δ CDA ≅ Δ CDB (SAS rule)

↓

AC = BC (congruent triangles have ≅ part)

↓

Δ ABC is isosceles (definition of isosceles)

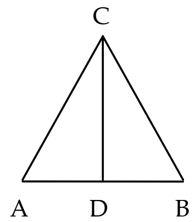


**Figure 1. Example of flow proof**

[16] To help students become good provers, sound criteria for judging the acceptability of their mathematical arguments are to be developed and successfully utilized in day-to-day teaching/learning (see Dreyfus, 1999).

### 2.2.4 Examining the use of items of mathematical knowledge in modelling the reality

Although the knowledge of geometry can support the solutions of many problems of everyday life such as building houses, packaging, advertizing, planning a sport field, constructing tunnels and bridges, and working out city and road maps (Graumann, 1989), it seems that many students are not aware of this important fact. To improve the matters, this activity can make use of three types of tasks, like those below that, respectively, deal with disclosing the underlying geometric knowledge, finding a suitable application of the given item(s) of this knowledge, and performing the complex application of geometric knowledge through modelling activities. Through utilizing these types of tasks[17], the knowledge of geometry will certainly become alive for the student, who will begin to perceive geometry as a human enterprise which improves our lives.

- The current position of a ship on the sea can be determined, among other methods, by measuring from it two horizontal angles defined by two of three objects at the coast that are represented on a map. This is because these objects and the obtained angles allow us to construct on the map two circles (see Figure 2), one intersection of which indicates the ship position (Gardner, 1987). Which item of knowledge enables this kind of navigation?
- A transversal intersects two parallel lines so that the alternate-interior angles are equal. Apply this item of knowledge to make an optical instrument. [For a solution, see Fremont (1979).]
- Analyze a water supply from a local lake. [A detailed analysis is presented in Matsumiya *et al.* (1989).]

[17] the first two of which may be more accessible to most students than the third

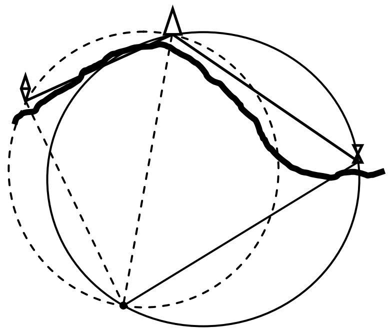

**Figure 2. Chart navigation: fix by horizontal angles**

## 2.3 Topic reconsideration

According to Niss (1996), we can distinguish three groups of mathematics education aims: utilitarian, disciplinary and personal, each of which can be recognized in the proposed four activities relevant to *any area of mathematical knowledge*. If we have in mind key-activities for doing and creating mathematics such as to calculate, apply, construct, play, evaluate, prove, order and find (Zimmermann, 2003), most of these are also covered by the examined activities.[18]

Although articles on the human face of mathematics are quite rare, many references deal with humanistic mathematics education[19], the two main aims of which may be promoting/establishing more links between mathematics and society and other sciences, and increasing mathematical

[18] Recall Freudenthal's (1991) approach "mathematics as a human activity" (p. 47), which calls for students' re-inventions of mathematics by using well-chosen practical problems relevant to their daily life.

[19] See, for example, articles published in *The Humanistic Mathematics Network Journal Online* available at www2.hmc.edu/www_common/hmnj/.

awareness concerning the applications of mathematics in society that today may be more relevant than technical skills (see Coray *et al*., 2003; pp. 91-92). Such an education would benefit from the proposed activities, the elaboration of which may involve other relevant issues. Some of them, suggested by Brown (1996), deal with:

- field boundaries (e.g. How may solved and unsolved problems limit the thinking in mathematics? How may the field of mathematics be different from other humanistic studies and experiences?), and
- personal preferences (e.g. How may people differ in posing problems, sharing problems, making problems into situations, etc.?).

Others concern the discovery/invention of the same piece of mathematical knowledge at different places and times[20], the role of truth in mathematics at different times and in different cultures, the relations between methods for inventions and methods for testing/proving (with respect to time and culture)[21], and the application of an ancient method to new domains[22]. Futhermore, important ideas from the history of mathematics, e.g. geometry as an encompassing tool for problem solving regarding pre-calculus methods, should be revitalized by means of technology [see Haapasalo and Stowasser (1993) or www.math.jyu.fi/~kahanpaa/TUBerlin/home.html]. However, despite their promising educational values, the need for design and use of such or similar activities has not been fully realized so far by most of the researchers in the field of mathematics

[20] For example, the method of Cavalieri was formulated already by the Chinese Zu Geng some 1200 years before Cavalieri (see Lam and Shen, 1985).
[21] See, for example, Zimmermann (2003).
[22] See Kadijevich (1993) where the method of *regula falsi* is successfully utilized in solving problems in arithmetic, differential equations, descriptive geometry and programming.

education.[23] Research may thus develop these activities further by making use of other relevant sources [see, for example, Singh (1998) and Mathematics Awareness activities at www.mathforum.com/mam/]. Research may also examine their empirical values in cognitive, metacognitive and affective terms. As a student's learning results from a complex interplay among his/her cognitive, metacognitive and affective domains — the last of which determines the global context where cognition takes place monitored and controlled by metacognition (see, for example, Schoenfeld, 1985) — such a study may primarily examine affective domain. Would teaching/learning based upon human face activities result in more self-confidence (cf. Eisenberg, 1991) or a higher mathematical self-concept[24]? Picker and Berry (2002) found that meeting with a diverse panel of mathematicians could change negative images about mathematics and mathematicians[25]. My experience with a generation of twelfth-year gymnasium students showed that writing matura (graduation) works on mathematical topics (all given within a humanistic context) can increase students' mathematical self-concept (Kadijevich, 2004c).

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[23] Tzanakis and Arcavi (2000) give concrete ideas and examples of integrating history of mathematics into mathematics education, but they only partly and implicitly provide material relevant to the presented four activities. The reader has probably visited the site *Mathematics with a Human Face* (see http://mathcentral.uregina.ca/HumanFace/), but we can find there just an approach focusing on mathematicians at work.

[24] This construct is examined in Opachich and Kadijevich (1997) and Kadijevich *et al*. (2003), for example.

[25] Eight mathematicians answered the questions of 179 seventh grade students about their work and lives.

# 3 Relating procedural and conceptual mathematical knowledge

## 3.1 Topic realization

Having examined Nesher (1986) and realized that mathematics education may not link procedural knowledge (skills) with conceptual knowledge (understanding), I became concerned about procedural and conceptual knowledge and their relation. On the basis of Skemp (1987), Tessmer *et al*. (1990), Vergnaud (1990) and Freudenthal (1978), I proposed that these knowledge types may be related directly (via objectification or proceduralization) or by means of inferential knowledge (Kadijevich, 1993). These ideas were presented at the 9th Congress of Yugoslav Mathematicians (in the section "Teaching, history and popularization of mathematics") in 1995 and summarized in the Serbian language in an article published by Mathematical Society of Serbia (Kadijević, 1995). In my PhD thesis I examined the acquisition and relation of the two knowledge types in problem solving through the development of expert system knowledge bases and found that such an approach can relate the two (Kadijevich, 1994).[26] As the need for a detailed account on

[26] The thesis used a sample of ninth-grade gymnasium (high school) students who developed these bases concerning problems on motion of one and two objects [see also Kadijevich (1998a, 1999, 2000) and Kadijevich and Haapasalo (2001)].

these knowledge types and their relation, especially in a technologically-supported learning environment, was there realized, further research concerning both theoretical and empirical/instructional issues was undertaken in the end of the 1990's with Professor Lenni Haapasalo at the Universities of Jyväskylä and Joensuu, Finland. The results of this joint research can be found in papers **I** and **V**. These papers are cited, for example, in Baker and Czarnocha (2002) and Gordon (2004), respectively.

## 3.2 Topic consideration

### 3.2.1 Distinction by terminology

Because of different research frameworks and the fact that procedural and conceptual knowledge are not easy to define precisely (Carpenter, 1986), a number of views relating to procedural vs. conceptual knowledge can be found in the literature. Having examined some twenty such views and chosen a dynamic view of conceptual knowledge, we (see **I**) put forward the following *P-C* distinction.

- *Procedural knowledge* denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation form(s). This usually requires not only knowledge of the objects being utilized, but also knowledge of format and syntax for the representational system(s) expressing them.
- *Conceptual knowledge* denotes knowledge of and a skilful "drive" around particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms.

Procedural knowledge often calls for automated and unconscious steps, whereas conceptual knowledge typically requires conscious thinking. However, the former may also be demonstrated in a reflective mode of thinking when, for example, the student skilfully combines two rules without knowing why they work.

### 3.2.2 Distinction of procedural and conceptual tasks

It is hard to develop conceptual (procedural) test items that are procedurally (conceptually) free as most items of knowledge have both conceptual and procedural features (Silver, 1986). Despite that, most empirical studies on procedural and conceptual knowledge to date have been based upon two sets of test items assessing the levels of these types of knowledge (e.g. Nesher, 1986; Byrnes and Wasik, 1991; Palmiter, 1991; cf. Shimizu, 1996).

Developing procedural and conceptual test items is a particularly complex enterprise in the problem solving area. However, these distinctive sets regarding problem solving may in some cases be developed by using procedural and conceptual tasks. While the former involve fully quantified objects requiring extensive and for many students meaningless computations, the latter, which require genuine understanding of the underlying domain, involve not (fully) quantified objects requiring very little computation. As an example of this task distinction, consider the following two problems on motion having an identical underlying structure.[27]

[27] Again, what kind of thinking process a particular task is calling for a student, depends very heavily as well on curricular context as on the quality and organisation of the learning process.

- *Procedural task* - A car and a truck started simultaneously from towns that are 150 km apart. After what time did they meet each other if their speeds were 80 km/h and 60 km/h, respectively?
- *Conceptual task* - A mountaineer started his trip in the morning arriving at a mountain house in the evening. Having spent the night there, the mountaineer started down the next morning by using the same trail. Is there a point on the trail where he was at the same place at the same time each day? Give a detailed explanation.

More conceptual tasks can be found in Dreyfus and Eisenberg (1990) and Kadijevich (1999a).

### 3.2.3 Searching for the relation between procedural and conceptual knowledge

How may this relation be basically supported by empirical data? Assume that the students' knowledge types were successfully assessed by two sets of *appropriate* test items (procedural vs. conceptual). Then one of the following seven relations between students' total scores on these knowledge types presented in Figure 3 can apply (while x denotes "many students", empty cell means "none or few")[28].

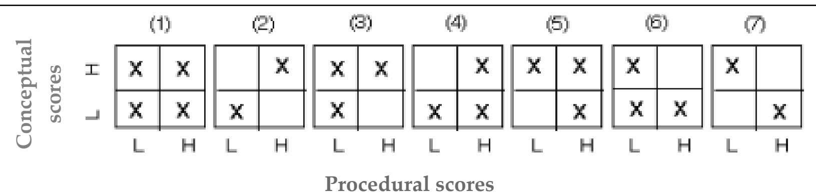


**Figure 3. Seven possible relations between test scores in conceptual and procedural mathematical knowledge**

[28] We (see **I**) display here possible, hypothetical outcomes supposing that each type of knowledge is demonstrated at *two* levels (low and high) defined, for example, with respect to the mean of the relevant score.

However, only the first four relations can be supported by existing theoretical and/or empirical research. They are characterized as follow.

1. *Inactivation view.* Procedural knowledge and conceptual knowledge are not related. Such a surprising outcome, which can hardly be theoretically defended, has been reported by Nesher (1986), and Resnick and Omanson (1987), for example.
2. *Simultaneous activation view.* Procedural knowledge is based upon conceptual knowledge and vice versa. This view can be recognized in Hiebert (1986), Byrnes and Wasik (1991) and Haapasalo (1993).
3. *Dynamic interaction view.* Procedural knowledge is based upon conceptual knowledge: *C* is a necessary but not sufficient condition for *P***.** This view was examined in Byrnes and Wasik (1991).
4. *Genetic view.* Conceptual knowledge is based upon procedural knowledge: *P* is a necessary but not sufficient condition for *C*. This view can be recognized in, for example, Kline (1980), Kitcher (1983), Vergnaud (1990), Gray and Tall (1993), and Sfard (1994).[29]

### 3.2.4 How may procedural and conceptual knowledge be linked?

An answer to this question seems to depend on whether one assumes the reliance of conceptual knowledge on procedural knowledge or vice versa. Many researchers find that procedural knowledge enables conceptual knowledge development. An instructional implication is: use procedural knowledge and reflect

[29] Good procedural knowledge can be demonstrated with missing or very limited conceptual knowledge (see Shimizu, 1996), which provides some implicit evidence for the genetic view.

on the outcome. We (see **I**) call this position the *developmental approach* as it reflects the developmental nature of mathematical knowledge, especially in early mathematics education.

Perhaps the majority of researchers/educators assume that conceptual knowledge enables procedural knowledge development. An instructional implication is: build meaning for procedural knowledge before mastering it. We (see **I**) call this position the *educational approach* since it seems to fulfil educational needs typically requiring a large body of knowledge to be understood, and to have supposed a transfer effect.[30]

Four answers to the raised question (yet not explicitly dealing with it) can be found in the literature. A summary of them is given below. It seems that the first two reflect the developmental approach, whereas the others assume the educational one.

Papert (1987) finds that *P-C* links are promoted through *microworlds coordination*. He assumes the learner's innate ability to divide the world (not a problem) into several (conceptual) *microworlds* enabling different procedures to be applied within each of them, (e.g. adding numbers in little worlds based upon finger manipulation, money facts and LOGO turtle geometry facts), claiming that it is basically the elaboration and coordination of these microworlds that enables conceptual knowledge to develop out of such fractured procedural knowledge.

Gray and Tall (1993) underline that it is *proceptual thinking* that enables linking procedural and conceptual knowledge. They

[30] Procedural knowledge does precede conceptual knowledge ontogenetically, but it is school learning that frequently precedes intellectual development (Vygotsky, 1978). Thus, for most topics, the educational approach may be more relevant than the other one. However, the utilization of an interplay of these approaches may, for some topics, be a better strategy than the application of one of them. An example of a sophisticated interaction between the two approaches can be found in Haapasalo (2003).

suppose the existence of *procept*, "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either of both" such as 'one half', 'fifty-fifty' and 1/2, claiming that procedural and conceptual knowledge are related through using "procedures where appropriate and symbols as manipulable objects where appropriate" (pp. 6, 8).

Anderson (1983) suggests that *P-C* links can be established through what may be called *production rules utilisation*. He finds that new task-specific productions (condition-action rules) have been initially developed through applying the available conceptual knowledge interpretively by means of some general problem-solving productions. These task-specific productions (i.e. newly generated skills) comprise procedural knowledge after knowledge *compilation* has been taken place. This compilation is based upon *composition* collapsing of a sequence of productions into a single production and *proceduralization* building versions of productions not requiring declarative knowledge retrieval (i.e. automating productions).

Gelman and Meck (1986) believe that it is *utilisation competence* that makes *P-C* links possible. They assume that this competence comprises various *enabling conditions*. In geometric tasks calling for locus constructions (e.g. construct a triangle, being given $a + b$, $c$, and $\beta$), an enabling condition (an utilisation competence item) is typically the following rule: "To determine a point that lies on a line with certain properties, construct this line obtaining a locus for that point".

### 3.2.5 Linking procedural and conceptual knowledge through CAL

If it is agreed that a main goal of mathematics education is to develop both procedural and conceptual knowledge and make links between the two, a very important research question is "how different technologies affect the relation between procedural and conceptual knowledge" (Kaput, 1992; p. 549). However, only a few CAL[31] studies have examined the effects of their treatments regarding the coordination of procedural and conceptual mathematical knowledge. While Schwarz *et al*. (1990) and Simmons and Cope (1997) found that the links between these knowledge types (the *P-C* links) can be established, Yerushalmy (1991), Hochfelsner and Kligner (1998) and Laborde (2000) found that their treatments did not promote any *P-C* links.[32]

Having in mind the presented four views on these links, they may be established through, for example, learning activities requiring production rules use and multiple representations transformation. These activities have been implemented in two constructivist CAL environments promoting *P-C*  While one implements production rules utilization through the development of expert system knowledge bases (Kadijevich, 1999), the other implements multiple representations transformation mostly through tasks on identification and production (Haapasalo, 1997). The presented empirical data regarding these environments prove that they can promote *P-C* links (see **V**).

[31] **C**omputer **A**ssisted **L**earning

[32] Most of these studies neither clearly define the relevant notions regarding the two knowledge types, measuring them reliably, nor thoughtfully examine the question of the *P-C* links at the theoretical and instructional levels.

## 3.3 Topic reconsideration

"Understanding how procedural knowledge and conceptual knowledge relate to one another is one of the major foci in mathematics education" (Shimizu, 1996; p. 234). Although the two cited studies of Haapasalo and Kadijevich (**I** and **V**) improve our understanding of the topic established by Hiebert (1986) some twenty years ago, more theoretical and empirical evidence on *P-C* links is needed because "the relation between computational expertise and conceptual understanding, and how each supports the other, is complex and requires careful study and thought" (Howe, 1998; p. 244).

Apart from developing tasks assessing the two knowledge types and their links, what other research directions may primarily be pursued in years to come?

Although I found that the applied qualitatively-oriented teaching promoted a gain in qualitative reasoning and related quantitative and qualitative reasoning, it did not promote a gain in quantitative reasoning (Kadijevich, 2002a), suggesting that the utilization of an appropriate sequencing or a combination of these kinds of teaching may be a better strategy than the application of just one of them. Thus, one important question is how procedures-first and concepts-first teaching/learning should be sequenced or combined in order to promote both the acquisition and coordination of procedural and conceptual knowledge. As the two knowledge types seem to develop iteratively (Rittle-Johnson and Koedinger, 2004) where a change of problem representation influences their relation (Rittle-Johnson *et al*., 2001), an appropriate solution to this question — recall that a general solution of any learning issue is rarely attainable in a constructivist sense — may benefit initially from the utilization of an interplay between the dynamic interaction and simultaneous activation approaches given in Haapasalo and Kadijevich (2004). As most mathematical objects can be viewed in several ways, they should, whenever possible, be examined from different perspectives, which suggests that *enabling/utilizing various learning paths* may be the main feature of learning environments making *P-C* links possible (Kadijevich, 2003).

Another important question regarding this topic is related to critical variable(s) influencing the *P-C* link. Baker and Czarnocha (2002) found that, although written mathematical thought (read metacognition and conceptual knowledge) was not dependent on procedural knowledge — no *P-C* links were found — these variables were related to the applied cognitive development measure, supporting its relevance to the *P-C* issues. As, because of their limited learning/thinking styles[33], some learners may tend to either proceduralize or conceptualize knowledge items, developing the two knowledge types and their links may be examined in the context of learners' learning/thinking styles (Kadijevich *et al*., 2003a). Such a proposal has been empirically supported by Kadijevich and Krnjaic (2004) where the higher field-independence the student demonstrated, the stronger *P-C* link he/she established.

Although it is very important to uncover how different technologies (i.e. learning opportunities of different technologies) affect *P-C* links, very little has been done in this area concerning

[33] It seems that individual's learning strategies (cognitive and metacognitive) are influenced and even constrained by some central strategies, such as holist and serialist originated from his/her learning style and approach to learning. The serialist strategy is used when the learner is rather concerned with details (usually in the order the material is presented), whereas the holist strategy is employed when he/she is more interested in the presented subjects as a whole, searching for important relations between ideas (Entwistle, 1988). For a distinction between axiomatic and relational thinking styles, see Williams (2002).

different technological environments, such as spreadsheets, dynamic geometry software, computer algebra systems and multimedia/hypermedia software. To improve the matters, the possibilities, limitations and unexploited issues of these environments are to be thoughtfully examined and utilized/ overcome/exploited (see Hvorecky, 2003; Haapasalo and Kadijevich, 2004; Kadijevich, 2002b, 2004a,b; Kadijevich and Haapasalo, 2004; Pesonen *et al*., 2002; and Siekkinen, 2003). Further research may address these questions promoting more appropriate learning activities relating the two knowledge types.

# 4 Utilizing mathematical modelling in a humanistic, technologically-supported way

## 4.1 Topic realization

Through examining an application-centred approach to mathematics education, several obstacles and neglected issues were identified (Kadijevich, 1993). These issues were presented at the 11th Conference on Applied Mathematics (a Yugoslav annual meeting) in 1996. Being mostly concerned with neglected issues, already presented in 1993 to Professor Morten Blomhøj, Roskilde University, I elaborated them in Kadijevich (1999b). Having in mind a broader humanistic context, I then incorporated them in the basic requirements for the design and assessment of a computer-based course on mathematical modelling (Kadijevich, 2003a). Realizing the tendency towards the standardization of technology-based mathematics education[34] and being aware that, contrary to most undergraduate courses with more or less known content and teaching method, such courses on modelling may (and probably do) differ considerably from institution to institution, I developed these requirements further in the form of standards of computer-based modelling. Aiming at a wider implementation of mathematical modelling in everyday teaching

[34] see the NCTM Principles and Standards for School Mathematics at http://standards.nctm.org/ and the ISTE Educational Technology Standards at www.cnets.iste.org

and learning of mathematics, I presented these standards at the ICMI Study 14 Conference (see **IV**). An elaboration of this contribution will appear in 2005 in the Study Volume (section on pedagogy edited by Professor Hans-Wolfgang Henn, University of Dortmund) published in the ICMI Study Series by Springer.

## 4.2 Topic consideration

Despite their unquestionable educational value, applications and mathematical modelling have so far had a marginal role in everyday teaching and learning of mathematics mostly (Blum *et al.*, 2002). To overcome this inappropriate state, we should help mathematics educators realize the full power of computer-based modelling, develop suitable standards of such a modelling, and ensure their proper utilization (see **IV**).

### 4.2.1 Realizing the full power of computer-based modelling

Even when computers are available, mathematics teachers rarely use them in their educational practice, probably because they do not have (enough) knowledge and skill related to what can be achieved by using these tools (see Manoucherhri, 1999). It seems that most mathematics educators do not realize the full power of computer-based modelling and, because of that, the wider inclusion of modelling in everyday mathematics education is simply not attainable. While modelling in general empowers a modeler's thinking and learning, computer-based modelling amplifies this empowerment through utilizing computers as versatile mindtools.

Many studies in mathematics and science education evidence that, instead of developing adequate mental models mirroring the presented conceptual models, students can often memorize the presented conceptual models and use them in school/academic settings, while they exploit their mental models in all informal settings [e.g. Vinner (1983) and Greca and Moreira (2000)]. Through explicitly-taught mathematical modelling (developing formal models and playing with already built such models), mental models can incrementally be developed in the direction of the desired conceptual models (see Greca and Moreira, 2000) as represented in Figure 4.

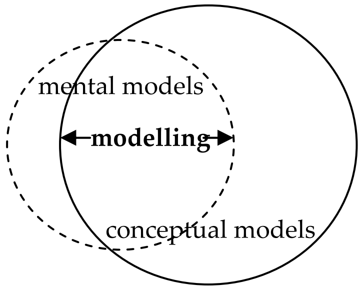


**Figure 4. From mental to conceptual models through modelling**

Despite the fact that the use of a sophisticated device and the transition from tool (impersonal device) to instrument (personal device) is achieved through a long process of instrumental genesis (Trouche, 2003), the use of computers as mindtools (Jonassen, 2000), expanding our mental function[35], can indeed be achieved in computer-supported modelling. Consider, for example, the utilization of Microsoft Excel and its various add-ins such as SimTools for simulations and iterative processes[36] and

[35] The Vygotskian view; it was Francis Bacon (1561-1626) who said "*Nec manus, nisi intellectus, sibi permissus, multum valent: instrumentis et auxiliis res perficitur*." — Left to themselves, neitheir hand nor intelligence is of much worth; the work is elaborated by using tools and aids. — taken from Ivic (1989; p. 430).
[36] see www.kellogg.nwu.edu/faculty/myerson/ftp/addins.htm

RISKOptimizer for simulation with optimization[37] enabling a very sophisticated modelling.

### 4.2.2 Developing suitable standards of computer-based modelling

Bearing in mind the outcome of educational research relevant to mathematical modelling[38], one may propose the following five standards of computer-based teaching of modelling relevant to upper secondary and tertiary levels of mathematics education.

- *Recognize a humanistically-oriented context of modelling*. Realize that, no matter how mathematically good a model may be, the applied data quantifications may be arbitrary, the selected optimization criteria subjective, and the chosen applications questionable. Be aware that the developed models just give decision makers additional information to help them become better informed, and that it is always a human who decides on the course of action and takes full responsibility for its consequences.
- *Present modelling as a complex process*. Deal with several incrementally-created models concerning the same real life situation.[39] Realize the complexity of modelling arising from an interplay among modelling steps and from interactions among modelling actors whose ways of thinking, values, attitudes, preferences etc. may be quite diverse. Be aware that the development of an institutionalized model may require 100 more times than the development of a prototype model.

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[37] see www.palisade-europe.com/html/risko.html

[38] See Burghes and Wood (1984), Ossimitz (1989), Lambert *et al*. (1989), Schoenfeld (1992), Ikeda (1997), Heugl (1997), Kadijevich (1999b), Greca and Moreira (2000), Galbraith and Haines (2001), paper **V**, Moore and Weatherford (2001), Blum *et al*. (2002), Galbraith (2002), and Niss (2003).

[39] Develop such models or examine such models developed by others.

- *Use modelling to empower thinking and learning.* Be aware of existing mental models and desired conceptual models. Help students realize the validity and possible limitations of their mental models. Help them incrementally improve their mental models towards the desired conceptual models through the development of appropriate mathematical models.[40] Use modelling to promote better understanding.
- *Recognize and empower cognitive, metacognitive and affective issues of modelling.* Be aware that modelling is based on a demanding interplay of modeler's cognitive, metacognitive and affective domains. Help students carry out required matematizations (clarify a real problem, generate variables, select variables, and set up conditions) confidently. Help them set up skilfully those conditions that enable an easy (easier) solution to the mathematical problem. Help them evaluate models critically.[41] Promote positive affective contexts about mathematics and the problem domain.
- *Use computers as mindtools for modelling.* Apply versatile tools such as Casio ClassPad[42], Microsoft Excel or Texas Instruments Derive. Avoid, whenever possible, promoting only the “black box” view of the applied tool[43], by giving/requiring conceptual

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[40] Requiring students to provide conceptual and procedural clarifications of the developed models, their application and the obtained outcomes can help teachers realize the differences between mental and conceptual models, manage the desired transition and eventually achieve it.

[41] As a next step, each of these *help* indicators should be clarified in the following way: “By applying … help students …” For example, “By assisting students realize basic features of appropriate models, help modelers evaluate the developed models critically.”

[42] ClassPad is a calculator but it can be emulated at a computer screen by using the ClassPad Manager software.

[43] Learning of mathematics should be based upon *the white box black box principle* (from explanations to routine applications) or *the black box white box principle* (from explorations to explanations), where a technological tool can be used as a

and procedural explanations of the performed actions and calculations. Require students to solve tasks involving routine calculations, conceptual conclusions and links between procedural and conceptual knowledge. View computers as tools that expand our mental function.

Like ISTE ET standards, each of these or other agreed standards may be described by a list of suitable indicators mirroring the sentences used in its initial description.

### 4.2.3 Ensuring a proper utilization of such standards

To avoid a discrepancy between intended and implemented standards of computer-based modelling, pre-service and in-service professional development of mathematics teachers should successfully deal with various critical issues. One of them is related to realizing the full power of computer-based modelling. The other three are summarized below.

- *Selecting basic indicators of the official standards.* Let us suppose that we utilize 8 modelling standards comprising 30 indicators. Many teachers, especially those less-experienced and not so technology-minded, may find these indicators quite demanding. Such teacher may thus initially base his/her teaching practice just upon several basic indicators, still bearing in mind the broader context[44]. Utilizing an opportunity to select one's own indicators is particularly valuable to those involved in professional teacher development when it focuses on issues that are subject to change (see Kadijevich, 2002c).

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"black box" either to put into practice previously learned knowledge and skills or to support explorations that will be followed by conceptual and procedural clarifications (see Heugl, 1997).

[44] possible extended by one's own sound indicator(s)

- *Making the selected indicators alive.* As learning through multimedia design can be beneficial to students in many ways resulting in better understanding of and more interest in mathematical, didactic and technological issues (see **VI**), (future) mathematics teachers should become designers of multimedia lessons.[45] Because the suggested/targeted set of professional/psychological/didactic guidelines may be too demanding to be implemented successfully[46], a multimedia project instructor should encourage project participants to choose their own subsets of these guidelines and help them implement these subsets successfully.
- *Reinforcing the context of the official standards.* Having in mind the advantages of Web-based professional development for mathematics teachers (Shotsberger, 1999), a critical, balanced and well-designed implementation of modelling standards may be achieved by such a support in an easier and/or a less time consuming way. Mathematical faculties and professional teacher organizations should thus support this kind of development and maintain Web sites where continuing computer-based modelling experiences are provided (adapted from Kadijevich, 2002c).[47]

[45] simply done through the development of HTML files (use Microsoft Word and its Save As Web Page command) comprising Java applets downloaded from the Internet

[46] "You get frightened when thinking of all these requirements!" – a student's reaction.

[47] Such a requirement is in accord with Kilpatrick (2003) who underlines that to improve the practice of mathematics teaching, we need "the creation of new forms of continuous professional [teacher] development" (p. 326).

## 4.3 Topic reconsideration

Although humanistic and technological perspectives on mathematics education may appear to be unrelated, positive connections between these perspectives can still be established. "If attention is paid to what mathematics is important for people to learn on an assumption of availability of technology, what computational tasks are safely and properly left to technology and what fresh opportunities for learning mathematics are provided by technology, we can make progress on restoring humans to their rightful place at the center of mathematics education, surely the main point of a humanistic renaissance." (Kissane, 2002; p. 195)

Research has been concerned with teaching/learning modelling in a humanistic, technology-supported way — see, for example, Molyneux-Hodgson *et al*. (1999) and Ferrucci and Carter (2003) — but, to my knowledge, no study thoughtfully integrates humanistic and technological issues to the degree presented here. It is true that, no matter which learning environment is utilized, many students experience difficulties in moving between the real and the mathematical world (see Crouch and Haines, 2004), but, despite different students' views of the utilized technology (potentially) promoting different kinds of learning (Galbraith, 2002), technology can help reduce such difficulties (see Keune and Nenning, 2003), enabling us to concentrate on subtasks causing the most difficulties in moving between the two worlds.

As regards modelling pedagogy, Blum *et al*. (2002) call for "appropriate pedagogical principles and strategies for the development of applications and modelling courses and their teaching" (p. 164). The presented standards can help us define and successfully utilize such principles and strategies. Researchers in the modelling community may thus focus on an elaboration of these standards involving issues of assessment and teacher's professional development as well as on their adjustments to different educational levels.[48] Research may also focus on critical variables influencing (future) utilization of such standards by elaborating, for example, the approach of Kadijevich *et al.*, (2004a) examining teachers' attitudes to achieve ET standards in terms of their computer attitudes and professional support to do so achieved during their pre-service teacher education.

[48] Although good modelling standards will not guarantee good modelling practice and expected educational outcomes, they would — primarily applied as a useful framework and not as a dogmatic recipe — confidentially help us spread the agreed modelling philosophy, recruit its followers among skilled and open-minded educators, manage their professional development, and assess the effects of educational outcomes enabling adequate further steps.

# 5 Promoting technology-based learning through applications and modelling, multimedia design, and on-line collaboration

## 5.1 Topic realization

Having examined Wilson and Cole (1991), I realized the potential benefits of learning through design, and examined them in my PhD thesis dealing with expert system knowledge base design (Kadijevich, 1994). During the realization of the course *Didactics of Informatics* at the Faculty of Mathematics, Belgrade University, in the academic years 2000/2001 and 2001/2002, I came across ISTE educational technology standards and introduced these to students (Kadijevich, 2002c). These standards, among others, require learners to utilize technology as a tool for communication, research, problem-solving, decision-making, and productivity. Searching for a didactically-relevant book supporting such requirements, I found Jonassen (2000) quite inspiring. Other references that have strengthened my interest in this topic and influenced my approach to it were Crowe and Zand (1997), Mayer (2001) and Moore and Weatherford (2001), the last two of which I have used in teaching courses on multimedia in education and basics of modelling, respectively. The consideration of this topic is based upon papers **III**, **VI** and **VII**.

## 5.2 Topic consideration

The three kinds of technology-based learning listed in the title of this chapter (learning through applications and modelling, learning through multimedia design, and learning through on-line collaboration) are considered in the following three sections. Although they can be examined and applied independently, they should be combined in suggested or other suitable ways as each of them can benefit from the other(s).

### 5.2.1 Learning through applications and modelling

To realize the educational values of such a technology-based learning emphasized in the previous chapter, let us consider the following two examples utilizing different kinds of technological tool (see **VII**).

By using ClassPad 300[49], a solution to a traffic jam problem — where a velocity and a separating car distance for an one-line tunnel road during peak hour traffic is to be found — can be obtained in the following way: determine a quadratic polynomial fitting the data comprising different velocities and corresponding total stopping, compose a function to be maximized, drag its expression and drop it to the Geometry window, and look for a velocity maximizing the traffic flow. The drag & drop activity is a basic ClassPad utility, which helps learners not only to realize how a change in the symbolic representation of an object affects its graphical representation and vice versa, but also to relate abstract concepts when their symbolic and geometric representations are examined together (the flow function is maximized where the graph of the first derivative of that function crosses the $x$-axis).

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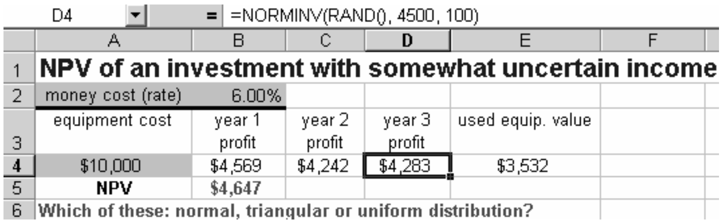
[49] see www.classpad.org/overview.html

When using Microsoft Excel[50], a solution to a task of finding the NPV (Net Present Value) of an investment when equipment cost and expected annual profits are known can be easily obtained by utilizing the NPV built-in function. Then we can gradually introduce refinements regarding money cost change, inflation (change) and forecasted profit uncertainty, the last of them can utilize the Excel built-in functions of statistical type (see Figure 5). Models in Excel can be, if need be, gradually refined through the use of optimization, simulation, iterative process, and simulation with optimization (see footnotes 36 and 37).



D4 = =NORMINV(RAND(), 4500, 100)

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | NPV of an investment with somewhat uncertain income | | | | | |
| 2 | money cost (rate) | 6.00% | | | | |
| 3 | equipment cost | year 1 profit | year 2 profit | year 3 profit | used equip. value | |
| 4 | $10,000 | $4,569 | $4,242 | $4,283 | $3,532 | |
| 5 | NPV | $4,647 | | | | |
| 6 | Which of these: normal, triangular or uniform distribution? | | | | | |

**Figure 5. Screenshot of an Excel model on the NPV of investment**

The examined learning, which may be utilized through on-line collaboration, should not be limited to a particular tool or technology. No matter which able technology is utilized, the following principle should apply: "When using mathematics, don't forget available tool(s); when utilizing a tool, don't forget the underlying mathematics". Applied mathematics can direct the tool utilization as the tool availability can suggest what mathematics may be utilized[51], promoting a true humanistic

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[50] see www.microsoft.com/office/excel/default.asp

[51] Although the position "the tool availability can suggest what mathematics may be utilized" may also have a negative aspect/effect (e.g. be focused on that mathematics that can be handled by the tool, which may not be appropriate to solve the problem adequately), it should primarily be viewed as the following requirement: "don't limit yourself to a particular tool or technology; if need be,

approach *Modello, ergo comprehendo* where the utilization of technology complements the learning of mathematics through carefully-designed learning activities respecting relevant cognitive, metacognitive and affective issues (see **VII**).

### 5.2.2 Learning through multimedia design

Multimedia is a powerful tool for knowledge construction. As those who learn more from the instructional materials are their developers, not users (Jonassen, 2000), (future) mathematics teachers should design multimedia lessons[52] and thus become knowledge constructors rather than knowledge users[53].

Multimedia lessons should be developed on sound *multimedia learning principles* (Mayer, 2001). As regards mathematics such principles may require the following: (1) develop multimedia lessons by at least combining words and pictures; (2) achieve a solid technical realization; (3) show the underlying mathematical structure of the chosen topic; (4) present its application(s)[54]; (5) enable various learning paths within it; and (6) deal with relevant procedural and conceptual mathematical knowledge and the links between the two.[55]

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search for a better one supporting more mathematical concepts and procedures".

[52] an appropriate place where the third kind of learning may be utilized

[53] Eskelinen (2004) found that the design of a hyper-text based software for the learning of measurement and accuracy promoted students' understanding not only of the underlying mathematics but also of the teaching and learning of this topic. His research doesn't support the traditional approach to teacher education where computer skills are taught separately from the knowledge structures and pedagogical thinking.

[54] an appropriate place where the first kind of learning should be utilized

[55] Principles (3) – (6) are supported by **I, II**, Kadijevich (2003), and Cukrowize and Zimmermann (2000-3). To make the project easier, the historical and epistemological issues of mathematical knowledge were not listed among the design requirements.

As a part of course *Didactics of Informatics* future mathematics teachers at the Faculty of Mathematics, Belgrade University, developed multimedia mathematical lessons (see **VI**). At the beginning of the project their teacher (the author of this report) explained the above-mentioned design principles to the students and listed some Web sites where suitable, applet-based lessons can be found.[56] The multimedia designers worked in 12 groups (mostly two students in a group), elaborating topics chosen by themselves. The project lasted five weeks and *no technical support* was given to the students.

These twelve groups developed multimedia HTML pages (i.e. simple hypermedia[57]) on various topics such as squaring a binomial; circle and angles; similarity; ellipse; derivative; and integral. Although different technical solutions were applied, most students based their lessons on Java applets, which were downloaded from the Internet and/or especially developed for this multimedia project.

Despite the fact that many of them initially lacked technical skills relevant to their pursued approach (using Java applets within Web pages; the work with Front Page, Flash, Java, Euklid, etc.), the designers developed multimedia lessons at a good technical level. Requirements (3) and (4) were much easier than (5), which was much easier than (6). Requirements (3) and (4) were more or less implemented in 11 developed artifacts. Some kinds of different learning paths were implemented in 8 artifacts (intuitive and theoretical approaches, two solutions of a task, two ways of introducing a concept). Only 2 artifacts tried to respect the P/C requirement (determining a position using the concept of hyperbola, procedural/conceptual quiz questions). However,

[56] see www.saab.org/moe/start.html and www.ies.co.jp/math/java/

[57] as most created lessons utilized several hyperlinks referred to the developed HTML files or their parts

having in mind the subjects' experience and relevant prior knowledge as well as the subtlety of the proceduralo-conceptual issues [see **I** and Haapasalo and Kadijevich (2004)], the project can be considered successful (for more detail see Kadijevich 2002d). Two screenshots concerning examples of animations for multimedia ellipse (one of the best artifacts developed by the students) are given in Figure 6.

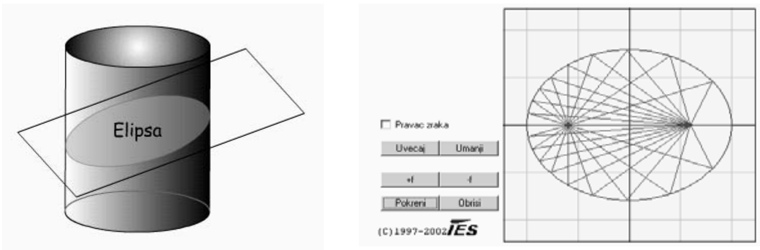


**Figure 6. Two screenshots of animations in a multimedia lesson on ellipse**

On the basis of Kadijevich (2002d), a Finnish study involving future mathematics teachers at the Faculty of Education, University of Joensuu, compared procedural with conceptual approach to multimedia design. While the procedural approach (play with a prototype and change it to achieve something more mathematical for pupils) caused more pedagogical discussions among the students, the conceptual approach (based upon a mini-lesson about knowledge of a Web page involving an applet) forced students to discuss the logical and technical structure of their Web pages. Despite these different design approaches (procedural vs. conceptual), most students expressed encouraging comments about pedagogical ideas gained through this project (see **VI**).

### 5.2.3 Learning through on-line collaboration

Innovative education can and should be mediated through the Internet (e.g. Hinostroza and Hepp, 1999).[58] As distance learning can be substantially improved with computer-mediated communication (Crowe and Zand, 1997), the learning of mathematics may be realized in an Internet-based collaborative environment, say ICELM[59], by using email correspondence. Having in mind ISTE educational technology standards for students, such an environment should utilize versatile software products as tools for communication, research, problem-solving, and productivity. To achieve this end, the environment may be built around a skilful multitasking with programs such as Internet Explorer, Word and Excel, as well as Derive and/or Cabri-geometry (www.ti.com), especially in upper secondary and tertiary education. Having familiarized themselves with the chosen programs, the teacher and his/her students may use the environment taking the following steps (see **III**):

- the teacher places a challenging task on his/her Web site and sends a note such as "see the task" to the students;
- the students visit the site and complete, within the prescribed time (usually 2–3 days), the proposed task in pairs[60], by using "e-talks" to each other[61];

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[58] Especially when we need to teach students at a faster pace, to support their learning how to learn or to guide them in collaborative learning, empowering group learning by mixing students with different thinking styles (see Lee and Tsai, 2004). Apart from social insecurity issues, possible limitations of this approach may primarily be found in the versatility of utilized Internet-based tools supporting effective complex thinking collaborations in problem solving, decision making and designing (see Jonassen, 2000).

[59] **I**nternet-based **C**ollaborative **E**nvironment for the **L**earning of **M**athematics

[60] or in groups of at most 3-4 students (to enable an effective communication)

[61] Instead of e-talks, the pairs may use the SKYPE software for telephone-discussion groups up to 4 participants via Internet (see www.skype.com) and, if

- having completed the task, the pairs summarize the outcome comprising textual and graphical data in form of a Web-presentation[62] generated by Word;
- the files are sent to the teacher who assesses the students' accomplishments;
- the teacher sends to the pairs (places on his/her Web site) some of the submitted outcomes and asks the pairs to review them and suggest how they may be elaborated in respect to their content and/or presentation;
- the reviewers produce, within the prescribed time (usually 2 days), reviews as suitable presentations, and send them to the teacher;
- having assessed the reviews, the teacher places on his/her Web site all submitted outcomes along with his/her remarks and the reviews' comments if any, ranks all pairs according to their accomplishments so far, and sends a note like "see the results" to the students.

These steps may be completed within a week. To achieve an effective communication based, for example, upon Schoenfeld's (1992) problem solving management, many pairs may need considerable guidance from the teacher. To organize this guidance properly, the pairs' e-talks, which should be included in the submitted outcomes, are to be traced. Having in mind that

the teacher requires, attach to the submitted solution (some of) the exchanged message (and perhaps the recorded conversation). However, being aware that constructive, social learning is better supported by asynchronous than synchronous conferencing (Jonassen, 2000), programs for asynchronous conferencing such as FirstClass (www.softarc.com) or BSCW (http://bscw.fit.fraunhofer.de/) may be better choice than those supporting synchronous conferencing. According to Järvelä and Häkkinen (2002), fostering higher stages of perspective taking in an important critical issue of asynchronous conferencing.

[62] a suitable place where the fist two kinds of learning should be utilized

computer experience and computer attitude are positively correlated (e.g. Kadijevich, 2000a), the teacher should also uncover students' attitudes toward computers and try to make them more positive (if need be), especially those relating to perceived control over computers and their perceived usefulness.

The ICELM conceptual framework is based upon a form of constructivism called social constructivism influenced by Vygotsky's (1978) theory of intellectual development. For this form supposing a falibilist epistemology of human knowledge, the metaphor for the mind is persons in conversation[63], whereas the model of the world is a socially constructed, shared world (Ernest, 1994). The ICELM approach should thus be viewed as a tool that can empower some important aspects of learning captured by such a kind of constructivism.

Having in mind paper **II** and Kadijevich (1999a), ninth-grade ICELM students may be given (introductory) tasks like those below.

- By using an Internet search, find out five facts from the history of mathematics relating to right-angled trigonometry. These items may be examined in terms of time, context and logic of discovery, the last of which is probably the most important historical datum. For each item, give a web-site address containing a relevant datum (and estimate its reliability by examining various sources and their coherence).
- It is known that a transversal intersects two parallel lines so that the alternate-interior angles are equal. Justify this property in several ways by using various items of geometric knowledge. Use the property in a practical task, such as making an optical instrument.
- By using arithmetic, graphic and algebraic means, find out different solutions for the procedural task given on page 18.
- Solve the conceptual task on the same page.

[63] For Minsky (1986), this metaphor is mental agents in conversation.

## 5.3 Topic reconsideration

As technology-based learning through applications and modelling should not be limited to particular tool or technology — the ability to utilize aids and tools in an appropriate way is an important mathematical competency[64] — a relevant question for further study deals with factors that influence student's preference for a particular tool (see Geiger *et al*., 2003). Bearing in mind the process of instrumental genesis (Trouche, 2003), research may establish the main stages of this process for a particular tool (a set of tools). Of course, these stages should, among other things, be examined in terms of students' views of the utilized technology (recall Galbraith, 2002). Another important question concerning this learning is, as already underlined, the influence of different technologies on *P-C* links. Since modern Geographic Information System (GIS)[65] is a powerful mindtool, research may also focus on modelling by GIS and study critical issues of its successful implementation (Jovanović and Kadijević, 2004).

Learning through multimedia design can be a rewarding learning experience not only for future teachers [see **VI** and Bari and Gagnon (2003)][66] but also for middle school students (Liu

[64] Other competencies are: mathematical thinking, problem tackling, modelling, reasoning, representing, symbol and formalism, and communicating (Niss, 2003a; Blomhøj and Jensen, 2003). The NCTM Standards recognize the following competences: problem solving, reasoning & proof, communication, connections and representation (NCTM, 2000).

[65] Geographic Information System

[66] This kind of design, like any type of instructional design, should primarily be based upon didactical analysis of the covered topic [see Marjanović (2004) and Marjanović and Kadijevich (2001)] enabling, if possible, the utilization of various learning paths (Kadijevich, 2003).

and Hsiao, 2002)[67] and thus deserves to be studied in more detail.[68] However, research on multimedia may be based upon problematic assumptions[69], which requires us to move from evaluation to theory-driven research (Moore *et al*., 2004). Research may also search for interactive multimedia features maximizing learning efficiency (see *Ibid*), studying, for example, the issues of procedural and conceptual mathematical knowledge represented by Java applets (Kadijevich, 2004b).

Learning through on-line collaboration involving the two other kinds of learning[70] enriches learning possibilities and, by comparing the pursued approaches of different learners, helps us realize good (best) ways to learn a topic. Such complex learning also takes care of learners' culture, enabling the development of alternative forms of distance education respecting cultural, linguistic and other relevant issues (see Arnold *et al*., 1996). Research may thus focus on educational values of this learning examined in terms of, for example, thinking style, metacognitive abilities, computer attitude and mathematical self-concept.

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[67] Bari and Gagnon (2003) present five-step design process applied in French, Religious studies and Sciences; Liu and Hsiao (2002) report the impact of design activities on learners' knowledge of design process, awareness of cognitive skills required by design, resource management strategy, and motivation for learning.

[68] especially for hypermedia enabling a move from knowledge hierarchies to knowledge networks relevant to mathematics learning (see Burton, 1999)

[69] Do the features of multimedia match that of the human mind? Can competences of self-regulation and explicit scaffolding be supported by hypermedia structure? Is multiple channel presentation superior to single channel presentation?

[70] Imagine that on-line collaboration aims at creating multimedia lessons that comprise technology-supported solutions of tasks on applications and modelling. Recall Jonassen (2000) who underlines that "complex thinking is most likely to occur in shared workspaces, where students are collaboratively planning and designing a product" (p. 242).

Research may also look for a model representing the complex dynamics of students' personal development through online collaboration (cf. Shotsberger, 1999). However, as in case of multimedia, more research is needed on the theoretical frameworks of Internet-based learning (Hill *et al.*, 2004). As regards the learning of mathematics, such frameworks may primarily aid the design of mathematical problems and comprehension modelling tools within a humanistic context (Nason and Woodruff, 2004). Further research may therefore focus on developing these design frameworks, the first of which may elaborate on tasks given in paper **II** and Kadijevich (1999a).

# 6 Looking back

Aiming to improve mathematics education, this summary dissertation, based upon four journal articles and three published conference papers, examined four topics that have so far been neglected by the majority of researchers in the field. These topics were:

- promoting the human face of mathematics;
- relating procedural and conceptual mathematical knowledge;
- utilizing mathematical modelling in a humanistic, technologically-supported way; and
- promoting technology-based learning through applications and modelling, multimedia design, and on-line collaboration.

Although each of these can be treated separately in teaching/learning/research, the dissertation presented various connections among them, suggesting the following global framework for their consideration (Figure 7).

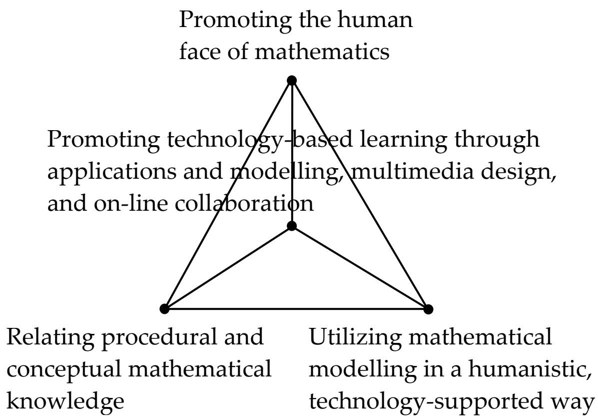


**Figure 7. Relations among the examined topics**

To establish a humanistic context of this research, the parts of the dissertation called *topic realization* briefly presented my motivation for examining the four topics and explained how the main ideas were realized and crystallized. The parts called *topic consideration* addressed the following questions:

- how the human face of mathematics may be promoted;
- how procedural and conceptual mathematical knowledge may and can be related;
- how a wider implementation of mathematical modelling in everyday teaching of mathematics may be attained; and
- how more opportunities for learning mathematics can be promoted by requiring students to do applications and modelling with versatile technological tools, to design multimedia/hypermedia lessons, and/or to create on-line collaborative works.

The parts of the dissertation called *topic reconsideration* gave a number of directions for further research including:

1. the refinement of the proposed learning activities promoting the human face of mathematics;
2. the impact of humanistic mathematics teaching/learning on students' mathematical self-concept;
3. critical variable(s) influencing the *P-C* link;
4. the impact of different technologies (i.e. learning opportunities of different technologies) on this link;
5. the refinement of the proposed standards of technology-based modelling;
6. critical variables explaining the utilization of technology-based modelling standards;
7. the main stages of instrumental genesis for a particular tool (a set of tools);
8. interactive multimedia features that maximize learning efficiency; and

9. the theoretical frameworks of Internet-based learning regarding the design of mathematical problems and comprehension modelling tools.

I hope that this work will justify the pursued approach and help us advance the field in years to come.

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Each of the hyperlinks given in this report was tested and found active as of October 30, 2004.